Zorn's Lemma Implies the Well Ordering Theorem

Proposition 0.1 *Zorn's Lemma implies the Well-Ordering Theorem. That is, assuming Zorn's Lemma, every set admits a well-ordering.*

Proof: Let V be an arbitrary set. Consider the collection

$$\mathcal{P} = \{(A, \preceq_A) : A \subseteq V,\ \preceq_A \text{ is a well-ordering of } A\}.$$

Define a relation $\sqsubseteq$ on $\mathcal{P}$ by

$$(A, \preceq_A) \sqsubseteq (B, \preceq_B)$$

if and only if the following three conditions hold:

1. $A \subseteq B$.
2. $\preceq_A \subseteq \preceq_B$.
3. For every $a \in A$ and $b \in B \setminus A$, we have $a \preceq_B b$.

In other words, A is an initial segment of B, and the well-ordering of B extends that of A.

We first verify that $\sqsubseteq$ is a partial order. Reflexivity is immediate. For antisymmetry, if

$$(A, \preceq_A) \sqsubseteq (B, \preceq_B) \quad \text{and} \quad (B, \preceq_B) \sqsubseteq (A, \preceq_A),$$

then the first two conditions give $A = B$ and $\preceq_A = \preceq_B$. For transitivity, suppose

$$(A, \preceq_A) \sqsubseteq (B, \preceq_B) \sqsubseteq (D, \preceq_D).$$

Clearly, $A \subseteq D$ and $\preceq_A \subseteq \preceq_D$. For any $a \in A$ and $d \in D \setminus A$, there are two possibilities. If $d \in B \setminus A$, then $a \preceq_B d$, and hence $a \preceq_D d$. If $d \in D \setminus B$, then $a \in B$ implies $a \preceq_D d$. Thus $(A, \preceq_A) \sqsubseteq (D, \preceq_D)$, establishing transitivity.

We next show that every chain in $\mathcal{P}$ has an upper bound in $\mathcal{P}$.

Let $\mathcal{C}$ be an arbitrary nonempty chain. Define

$$T = \bigcup_{(X,\preceq_X)\in\mathcal{C}} X, \qquad \preceq_T = \bigcup_{(X,\preceq_X)\in\mathcal{C}} \preceq_X .$$

We claim that $\preceq_T$ is a well-ordering of T.

First, $\preceq_T$ is a total order. Indeed, for any finitely many elements of T, there exists some $(X, \preceq_X) \in \mathcal{C}$ whose underlying set X contains all of them, because $\mathcal{C}$ is a chain. Moreover, the ordering on X agrees with the orderings of all smaller members of the chain. Consequently, reflexivity, antisymmetry, transitivity, and totality of $\preceq_T$ follow from the corresponding properties of $\preceq_X$.

It remains to prove that every nonempty subset of T has a least element.

Let U be an arbitrary nonempty subset of T. Choose $u \in U$. Since $u \in T$, there exists $(X, \preceq_X) \in \mathcal{C}$ such that $u \in X$. In particular, $U \cap X$ is nonempty. Since X is well-ordered, $U \cap X$ has a least element m under $\preceq_X$.

We claim that m is also the least element of U under $\preceq_T$. To see this, consider any $v \in U$.

If $v \in X$, then $m \preceq_X v$, and hence $m \preceq_T v$.

If $v \notin X$, choose $(Y, \preceq_Y) \in \mathcal{C}$ such that $v \in Y$. Since $\mathcal{C}$ is a chain, and $v \notin X$, we must have

$$(X, \preceq_X) \sqsubseteq (Y, \preceq_Y).$$

By the third condition in the definition of $\sqsubseteq$, we obtain $m \preceq_Y v$, and hence $m \preceq_T v$.

Therefore, m is the least element of U. This proves that $(T, \preceq_T) \in \mathcal{P}$.

Furthermore, for every $(X, \preceq_X) \in \mathcal{C}$, we have $X \subseteq T$ and $\preceq_X \subseteq \preceq_T$. The same argument as above shows that every element of $T \setminus X$ is greater than every element of X. Thus

$$(X, \preceq_X) \sqsubseteq (T, \preceq_T).$$

Hence $(T, \preceq_T)$ is an upper bound of $\mathcal{C}$.

The empty chain also has an upper bound, namely $(\varnothing, \varnothing)$. Therefore every chain in $\mathcal{P}$ has an upper bound.

By Zorn's Lemma, $\mathcal{P}$ has a maximal element $(S, \preceq_S)$.

We claim that $S = V$. Suppose otherwise. Choose $v \in V \setminus S$, and extend $\preceq_S$ to an ordering $\preceq'$ on $S \cup \{v\}$ by declaring v greater than every element of S. Explicitly, let

$$\preceq' = \preceq_S \ \cup \{(s, v) : s \in S\} \ \cup \ \{(v, v)\}.$$

This is a well-ordering of $S \cup \{v\}$: every nonempty subset intersecting S has a least element inherited from $\preceq_S$, while the only nonempty subset disjoint from S is $\{v\}$.

Moreover,

$$(S, \preceq_S) \sqsubseteq (S \cup \{v\}, \preceq'),$$

and these two elements of $\mathcal{P}$ are distinct. This contradicts the maximality of $(S, \preceq_S)$.

Therefore $S = V$, and $\preceq_S$ is a well-ordering of V.

Since V was arbitrary, every set can be well-ordered. $\square$