Laplace Transforms

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1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the Laplace Transforms Learning Guide*

ID	Learning Outcome	Achieved
1.	Explain why the Laplace transform is used in control engineering to convert ordinary differential equations (ODEs) from the time domain into the complex frequency domain.	☐
2.	Apply the Laplace transform to first- and second-order linear time-invariant (LTI) ODEs, assuming zero initial conditions.	☐
3.	Derive the Laplace transforms of step and ramp inputs, and use a table of transforms to obtain common Laplace transforms.	☐
4.	Rearrange the Laplace transform of an ODE into a transfer function and represent the transfer function in block diagram form.	☐

2.0 Introduction

In the ODE and Time Solution Learning Guide, ordinary differential equations (ODEs) were used to describe the dynamic behaviour of engineering systems such as an RC circuit, a water tank, a mass-spring-damper and a DC motor. The time solution of a first-order ODE was then obtained by solving the ODE

directly in the time domain, and the resulting expression was used to sketch the system response.

Solving ODEs directly works well for simple first-order systems, but it quickly becomes laborious for higher-order systems. It is also not well suited to a closed-loop control system, where several components (the controller, actuator, process and measurement device) are connected together and each is described by its own ODE. An important question therefore arises:

How can the dynamic behaviour of interconnected components be represented in a form that is convenient to manipulate?

In control engineering, the answer is the **transfer function**. A transfer function is an alternative and convenient representation of a linear time-invariant (LTI) ODE, and the mathematical tool used to move from an ODE to a transfer function is the **Laplace transform**. The Laplace transform converts an ODE into an algebraic equation, which can then be rearranged using the ordinary rules of algebra.

This Learning Guide shows how the Laplace transform is applied to first- and second-order ODEs, how the transformed equation is rearranged into a transfer function, and how the Laplace transforms of two common input signals, the step and the ramp, are obtained. These ideas form the foundation for block diagram algebra and for the inverse Laplace transform, which are developed in the Learning Guides that follow.

Historical Background

The Laplace transform method used by engineers today was originally known as Heaviside's operational calculus, after the English engineer Oliver Heaviside (1850–1925), who developed it to analyse electrical circuits. The method carries the name of the French mathematician Pierre-Simon Laplace (1749–1827), whose earlier work on integral transforms was later found to provide its rigorous mathematical basis. Until the late 1940s, control engineering developed largely independently in different countries (for example, the UK, the USA, Germany and the USSR). As these ideas were unified, the Laplace transform became one of the standard tools of control engineering.

Modelling in the Complex Frequency Domain

The Laplace transform maps a function of time, t, onto a function of the complex variable s, where:

$$s = \sigma + j\omega$$

Here, σ is the real part and ω is the imaginary part of s. The domain described by s is known as the complex frequency domain (or the s-domain). This domain is extremely powerful, both for understanding and for designing engineering systems that respond dynamically to known inputs and unknown disturbances. It is widely used in the design, refinement and implementation of closed-loop control systems, and later Learning Guides will show how it relates to the poles and zeros of a system and of its controller.

Why use the Laplace Transform?

The main reasons for using the Laplace transform in control engineering are:

i. It converts an ODE into an algebraic equation, which can be rearranged into transfer function form using the ordinary rules of algebra.
ii. Transfer functions can be represented as blocks and combined into block diagrams, allowing complete closed-loop control systems to be modelled. This is not convenient when working with ODEs directly.
iii. A table of transforms can be used to move from the time domain to the frequency domain (the Laplace transform) and back again (the inverse Laplace transform), without evaluating integrals each time.
iv. In the frequency domain, the transient and steady-state behaviour of a system can be determined more easily than in the time domain. The value of this will become clearer in later Learning Guides.

3.0 The Laplace Transform

The Laplace transform of a function $f(t)$ is denoted $\mathcal{L}\{f(t)\}$ and is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f\,(t)\, e^{-st}\, dt \tag{1}$$

where $\mathcal{L}$ denotes the Laplace transform, $f(t)$ is a function of time t, and $F(s)$ is the corresponding function of the complex variable s.

To find the Laplace transform of $f(t)$, the function is multiplied by e^{-st} and the product $f(t)e^{-st}$ is integrated with respect to t from $t = 0$ to $t = \infty$. For this integral to exist, the magnitude of the integrand must decay to zero as time increases, that is:

$$\lim_{t\to\infty} f(t)\, e^{-st} = 0 \tag{2}$$

The functions encountered in practical engineering systems satisfy this condition, and so they can all be Laplace transformed.

Stages of Transforming an ODE

Obtaining a transfer function from an ODE is a two-stage process:

- **Stage 1:** the Laplace transform is applied to each term of the ODE. This converts the ODE from the time domain into an algebraic equation in the complex frequency domain, in which the input, the output and the system appear as separate terms.
- **Stage 2:** the transformed equation from Stage 1 is rearranged algebraically into transfer function form, which is used extensively in control engineering.

Stage 1 requires the Laplace transforms of derivatives, which are derived next. Stage 2 is covered in Section 5.0.

Laplace Transform of a First Derivative

Let $y'(t)$ denote the first derivative of $y(t)$, that is, $y'(t) = \frac{dy(t)}{dt}$. Applying Equation (1) gives:

$$\mathcal{L}\{y'(t)\} = \int_0^\infty y'(t)\, e^{-st}\, dt$$

Integrating by parts, with e^{-st} differentiated and $y'(t)$ integrated, gives:

$$\mathcal{L}\{y'(t)\} = [e^{-st}\, y(t)]_0^\infty + s\int_0^\infty y(t)\, e^{-st}\, dt$$

The bracketed term is zero at the upper limit, by Equation (2), and equals $y(0)$ at the lower limit. The remaining integral is, by definition, $Y(s)$. Therefore:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) \tag{3}$$

where $y(0)$ is the initial condition. Throughout this Learning Guide, **zero initial conditions** are assumed, that is, the system starts at rest with $y(0) = 0$. In this case, Equation (3) simplifies to $\mathcal{L}\{y'(t)\} = sY(s)$. In other words, differentiating in the time domain corresponds to multiplying by s in the frequency domain.

Example 1: Laplace transform of a first-order ODE

Assuming zero initial conditions, consider the following first-order ODE:

$$\frac{dy(t)}{dt} + 2y(t) = u(t)$$

Undertake the Laplace transformation.

Solution:

With zero initial conditions, Equation (3) gives:

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s)$$

The Laplace transform is applied to each term of the ODE, with $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{u(t)\} = U(s)$, to give:

$$sY(s) + 2Y(s) = U(s)$$

Taking $Y(s)$ outside a bracket gives:

$$(s+2)\, Y(s) = U(s)$$

Laplace Transform of a Second Derivative

Applying the same process twice gives the Laplace transform of the second derivative:

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\, y(0) - y'(0) \tag{4}$$

With zero initial conditions, $y(0) = y'(0) = 0$, and Equation (4) simplifies to $\mathcal{L}\{y''(t)\} = s^2Y(s)$.

Example 2: Laplace transform of a second-order ODE

Assuming zero initial conditions, consider the following second-order ODE:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 4y(t) = 4u(t)$$

Undertake the Laplace transformation.

Solution:

With zero initial conditions, Equations (3) and (4) give:

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s) \qquad \text{and} \qquad \mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s)$$

The Laplace transform is applied to each term of the ODE to give:

$$s^2Y(s) + 3sY(s) + 4Y(s) = 4U(s)$$

Taking $Y(s)$ outside a bracket gives:

$$(s^2 + 3s + 4)Y(s) = 4U(s)$$

General Form

The general result for the nth derivative of $y(t)$ is:

$$\mathcal{L}\{y^{(n)}(t)\} = s^nY(s) - \sum_{k=1}^{n} s^{n-k}\, y^{(k-1)}(0) \tag{5}$$

where $y^{(k-1)}(0)$ is the value of the $(k-1)$th derivative of $y(t)$ at $t = 0$. With zero initial conditions, every term in the summation is zero, and so $\mathcal{L}\{y^{(n)}(t)\} = s^nY(s)$. Setting $n = 1$ and $n = 2$ recovers Equations (3) and (4).

4.0 Laplace Transforms of Common Inputs

To analyse how a system responds, a known input signal is applied to it. Two of the most common test inputs in control engineering are the step and the ramp. Their Laplace transforms are derived below from Equation (1).

Step Function

A **step function** (also known as the Heaviside step function) changes instantaneously from zero to a constant value a at $t = 0$ and then remains constant, that is, $f(t) = a$ for $t \geq 0$. A step input could represent, for example, a voltage being switched on or a new temperature set-point being requested. Applying Equation (1) gives:

$$\mathcal{L}\{a\} = \int_0^\infty a\ e^{-st}\,dt = a\int_0^\infty e^{-st}\ dt \tag{6}$$

since the constant a can be taken outside the integral. The integral of e^{-st} with respect to t is:

$$\int e^{-st}\,dt = -\frac{1}{s}\,e^{-st} \tag{7}$$

The limits of integration are then applied:

$$\int_0^\infty e^{-st}\ dt = \left[-\frac{1}{s}\,e^{-st}\right]_0^\infty \tag{8}$$

- **Upper limit,** $t \to \infty$: for $s > 0$ (more precisely, for $\sigma > 0$), $e^{-st} \to 0$.
- **Lower limit,** $t = 0$: $e^{-s(0)} = e^0 = 1$.

Substituting the limits into Equation (8) gives:

$$\left[-\frac{1}{s}\,e^{-st}\right]_0^\infty = -\frac{1}{s}\,(0-1) = \frac{1}{s} \tag{9}$$

Returning to Equation (6), where a was taken outside the integral, the Laplace transform of a step of magnitude a is:

$$\mathcal{L}\{a\} = \frac{a}{s} \tag{10}$$

When $a = 1$, the input is known as a **unit step**, with Laplace transform $1/s$.

Ramp Function

A **ramp function** increases linearly with time. A ramp of slope a is expressed as $f(t) = at$ for $t \geq 0$, and the unit ramp ($a = 1$) is simply $f(t) = t$. A ramp input could represent, for example, a set-point that increases at a constant rate. For the unit ramp, Equation (1) gives:

$$\mathcal{L}\{t\} = \int_0^\infty t\ e^{-st}\,dt \tag{11}$$

This integral is evaluated using integration by parts, $\int u\,dv = uv - \int v\,du$, where:

$$u = t\ \Rightarrow\ du = dt \qquad \text{and} \qquad dv = e^{-st}\,dt\ \Rightarrow\ v = -\frac{1}{s}\,e^{-st}$$

Applying integration by parts to Equation (11) gives:

$$\mathcal{L}\{t\} = \left[t\left(-\frac{1}{s}\,e^{-st}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\ dt \tag{12}$$

The boundary term in Equation (12) is evaluated first:

- **Upper limit,** $t \to \infty$: for $s > 0$, the exponential decays faster than t grows, so $t\,e^{-st} \to 0$.
- **Lower limit,** $t = 0$: the term is multiplied by $t = 0$, and so it is also zero.

The boundary term is therefore zero. The remaining integral in Equation (12) was evaluated for the step function in Equation (9), where $\int_0^\infty e^{-st}\,dt = 1/s$. Hence:

$$\mathcal{L}\{t\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2} \tag{13}$$

This is the Laplace transform of the **unit ramp**. For a ramp of slope a, the constant is taken outside the integral as before, giving $\mathcal{L}\{at\} = a/s^2$.

Table of Transforms

It is convenient to collect common Laplace transforms in a **table of transforms**, as in Table 2. A table of transforms is used to convert from the time domain to the frequency domain (the Laplace transform) and, working backwards, from the frequency domain to the time domain (the inverse Laplace transform). The table is extended, and the inverse Laplace transform is covered, in a later Learning Guide.

Table 2: *Laplace Transforms of Derivatives and Common Inputs*

Function	**Time domain,** $f(t)$	**Frequency domain,** $F(s) = \mathcal{L}\{f(t)\}$
First derivative	$\frac{dy(t)}{dt}$	$sY(s) - y(0)$
Second derivative	$\frac{d^2y(t)}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
General form	$\frac{d^ny(t)}{dt^n}$	$s^nY(s) - \sum_{k=1}^{n} s^{n-k}\; y^{(k-1)}(0)$
Step of magnitude a	a	$\frac{a}{s}$
Ramp of slope a	at	$\frac{a}{s^2}$

With zero initial conditions, the three derivative entries reduce to $sY(s)$, $s^2Y(s)$ and $s^nY(s)$ respectively.

5.0 Transfer Functions

The Laplace transform reduces a linear ODE to an algebraic equation, which can then be rearranged into transfer function form (Stage 2). The **transfer**

function of a system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input, assuming zero initial conditions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\text{output}}{\text{input}} \tag{14}$$

Equation (14) is a linear mapping from the input to the output. Rearranging it as $Y(s) = G(s)U(s)$ shows that the output is obtained by simply multiplying the input by the transfer function. This is represented in block diagram form in Figure 1, where the block contains the transfer function and the arrows represent the input and output signals.

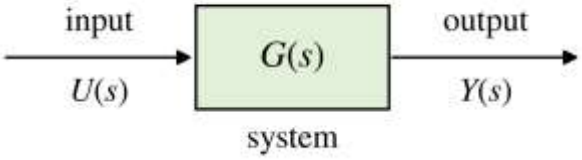


Figure 1: *Transfer Function Represented in Block Diagram Form*

The transfer function representation is not limited to the process being controlled. As shown in the ODE and Time Solution Learning Guide, other components of a closed-loop control system, such as a low-pass filter (RC circuit), an actuator and the controller itself, can also be described by ODEs and therefore by transfer functions.

Example 3: Transfer function of a first-order system

From Example 1, the Laplace transform of the first-order ODE was:

$$(s+2)\,Y(s) = U(s)$$

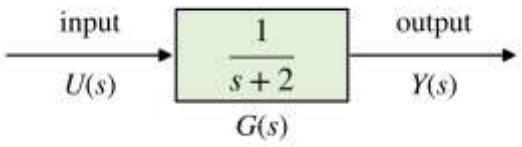
Convert this into transfer function form and represent it in block diagram form.

Solution:

Dividing both sides by $(s+2)$ and by $U(s)$ gives the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+2}$$

This is represented in block diagram form as follows:



Example 4: Transfer function of a second-order system

From Example 2, the Laplace transform of the second-order ODE was:

$$(s^2 + 3s + 4)Y(s) = 4U(s)$$

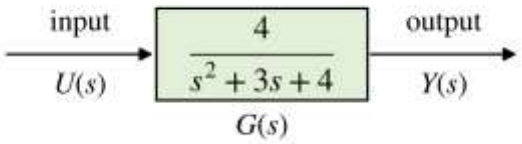
Convert this into transfer function form and represent it in block diagram form.

Solution:

Dividing both sides by $(s^2 + 3s + 4)$ and by $U(s)$ gives the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4}{s^2 + 3s + 4}$$

This is represented in block diagram form as follows:



The same two stages apply to any engineering system described by an LTI ODE. Example 5 returns to the RC low-pass filter that introduced the ODE and Time Solution Learning Guide.

Example 5: Transfer function of an RC low-pass filter

The RC low-pass filter is described by the following ODE, where $V_{in}(t)$ is the input voltage and $V_{out}(t)$ is the capacitor (output) voltage:

$$RC\,\frac{dV_{out}(t)}{dt} + V_{out}(t) = V_{in}(t)$$

Assuming zero initial conditions, determine the transfer function $H(s) = V_{out}(s)/V_{in}(s)$ and represent it in block diagram form.

Solution:

Stage 1: applying the Laplace transform to each term, with $\mathcal{L}\{dV_{out}(t)/dt\} = sV_{out}(s)$, gives:

$$RCs\,V_{out}(s) + V_{out}(s) = V_{in}(s)$$

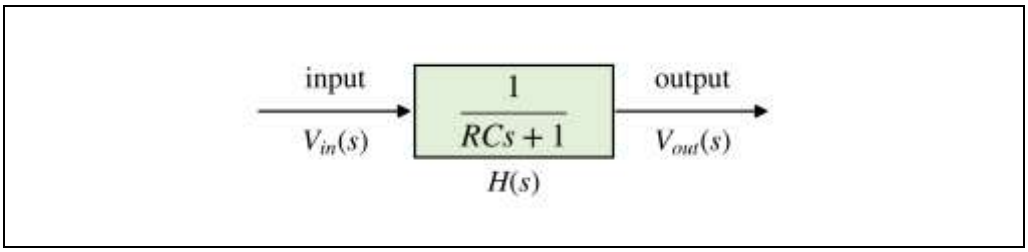
Taking $V_{out}(s)$ outside a bracket gives:

$$(RCs + 1)\,V_{out}(s) = V_{in}(s)$$

Stage 2: rearranging into transfer function form gives:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1}$$

Since $\tau = RC$ is the time constant, this can also be written as $H(s) = 1/(\tau s + 1)$, which is the standard form of a first-order transfer function. In block diagram form:



6.0 Exercises

Consider the following linear time-invariant (LTI) ordinary differential equations (ODEs), and assume zero initial conditions:

a) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = 5u(t)$

b) $2\frac{dy(t)}{dt} + y(t) = 2u(t)$

c) $\frac{d^2y(t)}{dt^2} + 2y(t) = 2u(t)$

d) $3\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} = 5u(t)$

e) $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 2u(t)$

f) $5\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = u(t)$

For each of the ODEs, undertake the following tasks:

i. determine the transfer function $G(s) = Y(s)/U(s)$
ii. represent the transfer function in block diagram form

Fully worked solutions to the above exercises can be found in **Appendix A**. A step-by-step video solution for Question a can be found on YouTube.

Once you are comfortable with the above, convert the ODEs of the engineering systems in Figure 2 (reproduced from the ODE and Time Solution Learning Guide) into transfer function form, and represent each transfer function in block diagram form.

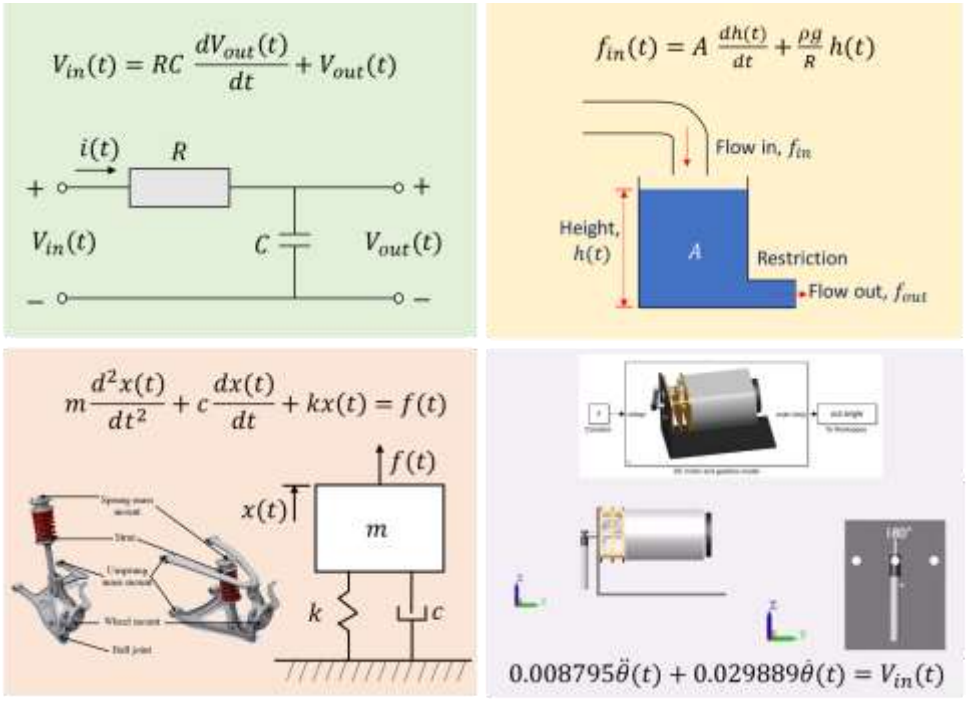


Figure 2: *ODEs of an RC Circuit (Top-Left), Water Tank (Top-Right), Suspension System (Bottom-Left) and DC Motor (Bottom-Right)*

Solutions to these exercises can be found in **Appendix B**. A step-by-step video solution can also be found on YouTube.

7.0 Summary

This Learning Guide introduced the Laplace transform as the mathematical tool used to convert a linear time-invariant (LTI) ordinary differential equation (ODE) from the time domain into the complex frequency domain. The benefits of working in the frequency domain were outlined: an ODE becomes an algebraic equation, which can be rearranged using the ordinary rules of algebra.

The Laplace transforms of first and second derivatives, and their general form, were derived and applied to first- and second-order ODEs. The transformed equations were then rearranged into transfer function form, $G(s) = Y(s)/U(s)$, and represented in block diagram form. The Laplace transforms of the step and ramp inputs were also derived, and the results were collected in a table of transforms.

For the continuous-time modelling of a closed-loop control system, each component (for example, the actuator, the process, a low-pass filter and the controller) can be represented by a transfer function. The key progression introduced in this guide can be summarised as:

ODE → Laplace transform → transfer function → block diagram

Transfer functions, together with the step and ramp inputs, are the starting point for the Block Diagram Algebra Learning Guide, where individual blocks are combined to model complete open-loop and closed-loop control systems.

Appendix A: Fully Worked Solutions to Exercises

For every question, the same two rules are used (with zero initial conditions):

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s) \qquad \text{and} \qquad \mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s)$$

Question a: $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = 5u(t)$

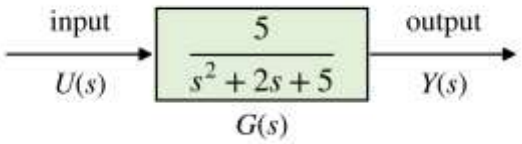
i. Applying the Laplace transform gives:

$$s^2Y(s) + 2sY(s) + 5Y(s) = 5U(s) \quad \Rightarrow \quad (s^2 + 2s + 5)Y(s) = 5U(s)$$

Rearranging into transfer function form gives:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5}{s^2 + 2s + 5}$$

ii.



Question b: $2\frac{dy(t)}{dt} + y(t) = 2u(t)$

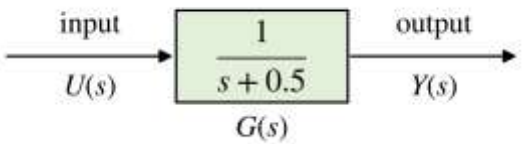
i. Applying the Laplace transform gives:

$$2sY(s) + Y(s) = 2U(s) \quad \Rightarrow \quad (2s + 1)\,Y(s) = 2U(s)$$

Rearranging into transfer function form, and dividing the numerator and denominator by 2 so that the leading coefficient of the denominator is 1, gives:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{2s + 1} = \frac{1}{s + 0.5}$$

ii.



Question c: $\frac{d^2y(t)}{dt^2} + 2y(t) = 2u(t)$

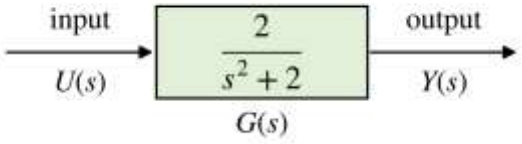
i. Applying the Laplace transform gives:

$$s^2Y(s) + 2Y(s) = 2U(s) \quad \Rightarrow \quad (s^2 + 2)Y(s) = 2U(s)$$

Rearranging into transfer function form gives:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 2}$$

ii.



Question d: $3\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} = 5u(t)$

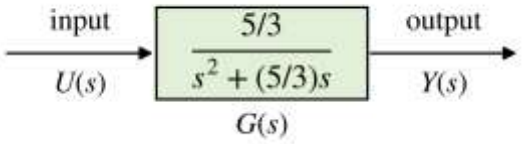
i. Applying the Laplace transform gives:

$$3s^2Y(s) + 5sY(s) = 5U(s) \quad \Rightarrow \quad (3s^2 + 5s)Y(s) = 5U(s)$$

Rearranging into transfer function form, and dividing the numerator and denominator by 3, gives:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5}{3s^2 + 5s} = \frac{5/3}{s^2 + (5/3)s}$$

ii.



Question e: $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 2u(t)$

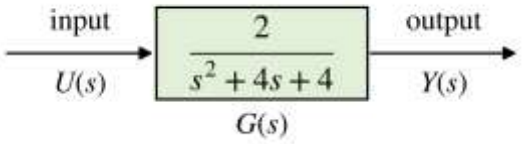
i. Applying the Laplace transform gives:

$$s^2Y(s) + 4sY(s) + 4Y(s) = 2U(s) \quad \Rightarrow \quad (s^2 + 4s + 4)Y(s) = 2U(s)$$

Rearranging into transfer function form gives:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 4s + 4}$$

ii.



Question f: $5\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = u(t)$

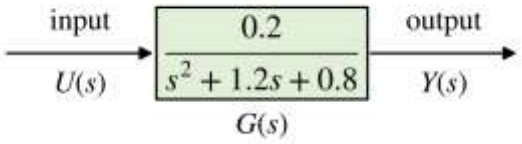
i. Applying the Laplace transform gives:

$$5s^2Y(s) + 6sY(s) + 4Y(s) = U(s) \quad \Rightarrow \quad (5s^2 + 6s + 4)Y(s) = U(s)$$

Rearranging into transfer function form, and dividing the numerator and denominator by 5, gives:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{5s^2 + 6s + 4} = \frac{0.2}{s^2 + 1.2s + 0.8}$$

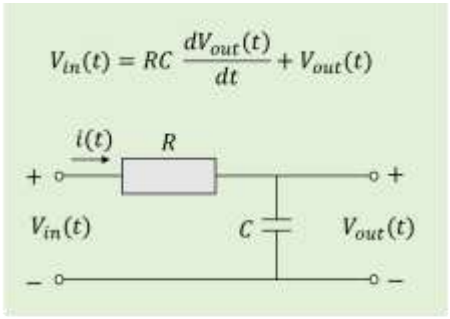
ii.



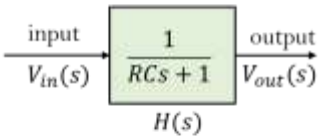
Appendix B: Solutions for the Engineering Systems

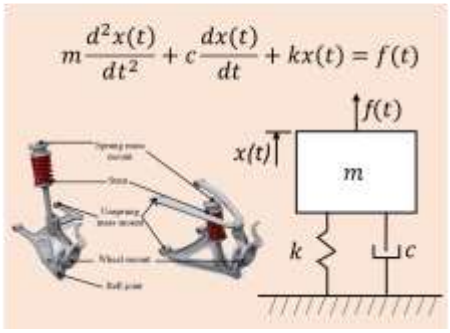
The transfer functions and block diagrams of the engineering systems in Figure 2 are given below.

RC circuit and suspension system: both follow directly from their ODEs.

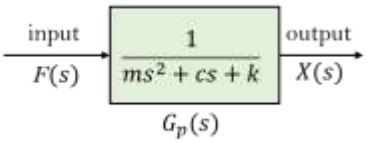


$$H(s)(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs+1}$$



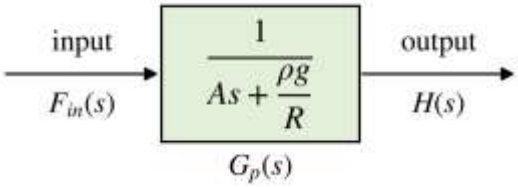


$$G_p(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k}$$

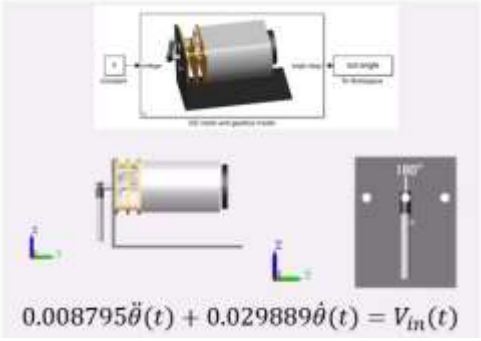


Water tank: applying the Laplace transform to $f_{in}(t) = A\frac{dh(t)}{dt} + \frac{\rho g}{R}h(t)$ gives $F_{in}(s) = \left(As + \frac{\rho g}{R}\right)H(s)$, and therefore:

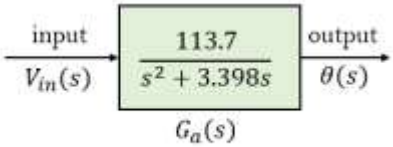
$$G_p(s) = \frac{H(s)}{F_{in}(s)} = \frac{1}{As + \frac{\rho g}{R}}$$



DC motor: dividing the ODE throughout by 0.008795 gives $\ddot{\theta}(t) + 3.398\,\dot{\theta}(t) = 113.7\,V_{in}(t)$, and so the transfer function between the input voltage and the motor angle is:



$$G_a(s) = \frac{\theta(s)}{V_{in}(s)} = \frac{113.7}{s^2+3.398s}$$



Recommended Citation

If you have used this learning guide in your teaching, learning or research, please cite it using the reference below, adapting the format as required by your preferred referencing style.

Pickering, J. E. (2026). Laplace Transforms. ACE-Lab. Teaching Notes: ACE-Lab Learning Guide, Version 1.0. https://zenodo.org/records/23301697