Inverse Laplace Transform (Time Solution)

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1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the Inverse Laplace Transform (Time-Solution) Learning Guide*

ID	Learning Outcome	Achieved
1.	Describe the stages involved in obtaining the time solution of an open-loop system or closed-loop control system transfer function.	☐
2.	Use the quadratic formula to determine the poles of a second-order transfer function and classify them as equal real, different real or complex.	☐
3.	Use partial fractions to express a transfer function in a form that allows the table of transforms to be used.	☐
4.	Apply the inverse Laplace transform to determine and sketch the time solution of a system subject to a step input, and verify the result using MATLAB.	☐

2.0 Introduction

In the Block Diagram Algebra Learning Guide, the transfer functions of individual components were combined to give the transfer function of an open-

loop system and of a closed-loop control system. These transfer functions describe the system in the complex frequency domain. However, the behaviour that would be observed in practice, such as how quickly a motor reaches its target angle, happens in the time domain. An important question therefore arises:

How can the response of a system in the time domain be obtained from its transfer function?

The answer is the inverse Laplace transform, denoted $\mathcal{L}^{-1}$. It converts a function of s back into a function of time, and so it gives the time solution of a transfer function when a known input is applied. The inverse Laplace transform uses the table of transforms, introduced in the Laplace Transforms Learning Guide, in reverse. Because the table only contains common functions, the expression to be inverted often has to be broken down first into simpler terms using partial fractions.

This Learning Guide sets out the stages involved in obtaining a time solution, extends the table of transforms, and shows how the quadratic formula and partial fractions are used to prepare a transfer function for the table. A complete example is then worked through by hand and checked using MATLAB.

Why use the Inverse Laplace Transform?

The inverse Laplace transform is important for both open-loop systems and closed-loop control systems because:

i. Modelling in the frequency domain is convenient. For example, a change to the design of a closed-loop control system can be made easily in the frequency domain, and its effect then evaluated using the time solution.
ii. The time solution can be determined in a small number of structured stages, and it provides information about the system response, such as its steady-state error and dynamic performance (covered in later Learning Guides).
iii. The same model can be simulated in MATLAB and Simulink to determine the time solution (the system response), which is especially useful when the mathematics becomes complicated.

3.0 Stages of the Inverse Laplace Transform

Figure 1 shows the ten stages involved in obtaining the time solution of an open-loop system or a closed-loop control system. The stages move from the time domain (top), into the complex frequency domain (bottom), and back to the time domain:

- **Stage 1** identifies the engineering problem: the open-loop system or closed-loop control system to be modelled and evaluated.
- **Stages 2 to 5** model the system: the ODE of each component is written (Stage 2) and converted into a transfer function using the table of transforms (Stages 3 to 5). These stages were covered in the ODE and Time Solution and Laplace Transforms Learning Guides.

- **Stage 6** uses block diagram algebra to combine the transfer functions, as covered in the Block Diagram Algebra Learning Guide.
- **Stage 7** applies a single input or reference, such as a unit step, to give the output $Y(s)$.
- **Stage 8** uses partial fractions, where needed. Often the output $Y(s)$ does not match any entry in the table of transforms directly, and so it is broken down into smaller fractions that do.
- **Stage 9** uses the table of transforms to undertake the inverse Laplace transform, converting each term from the frequency domain back to the time domain.
- **Stage 10** gives the time solution, $y(t)$.

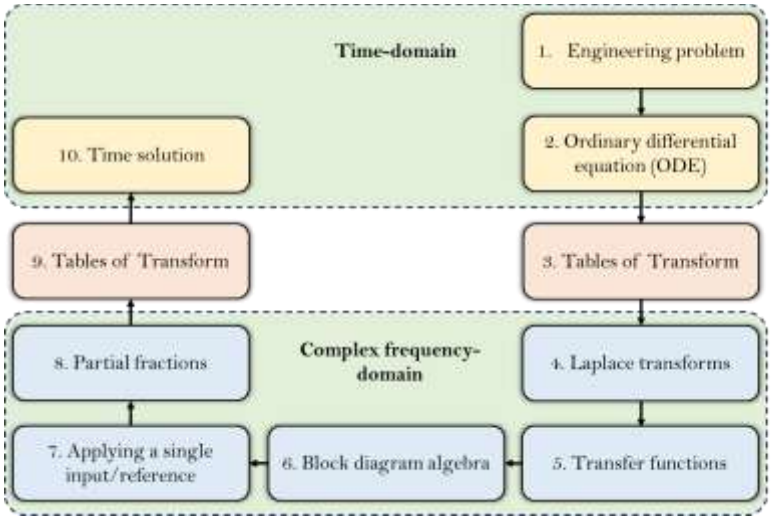
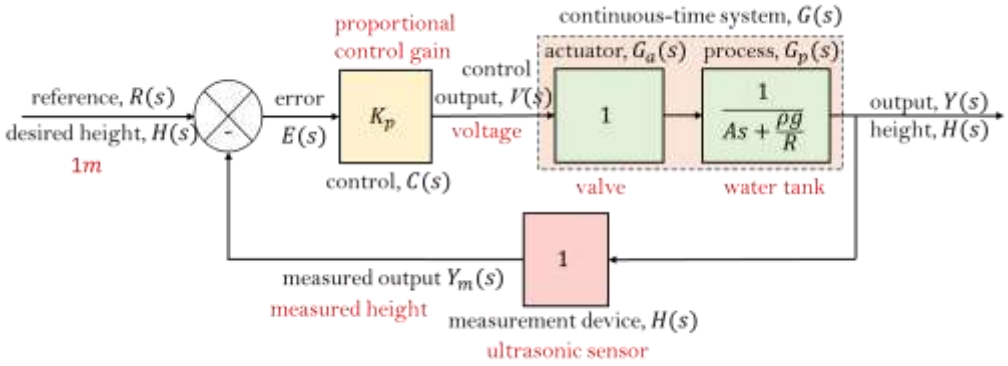


Figure 1: *The Ten Stages Involved in Obtaining the Time Solution of an Open-Loop System Transfer Function or a Closed-Loop Control System Transfer Function*

Engineering Problems (Stage 1)

Previous Learning Guides introduced several engineering problems that require feedback, and hence the design of a closed-loop control system. The closed-loop control systems for these problems (height, angle and displacement control) are shown in Figure 2. Each component has already been modelled as a transfer function (Stages 1 to 5), and so Stages 6 to 10 are needed to obtain the time solution. These systems are used as exercises at the end of this guide.



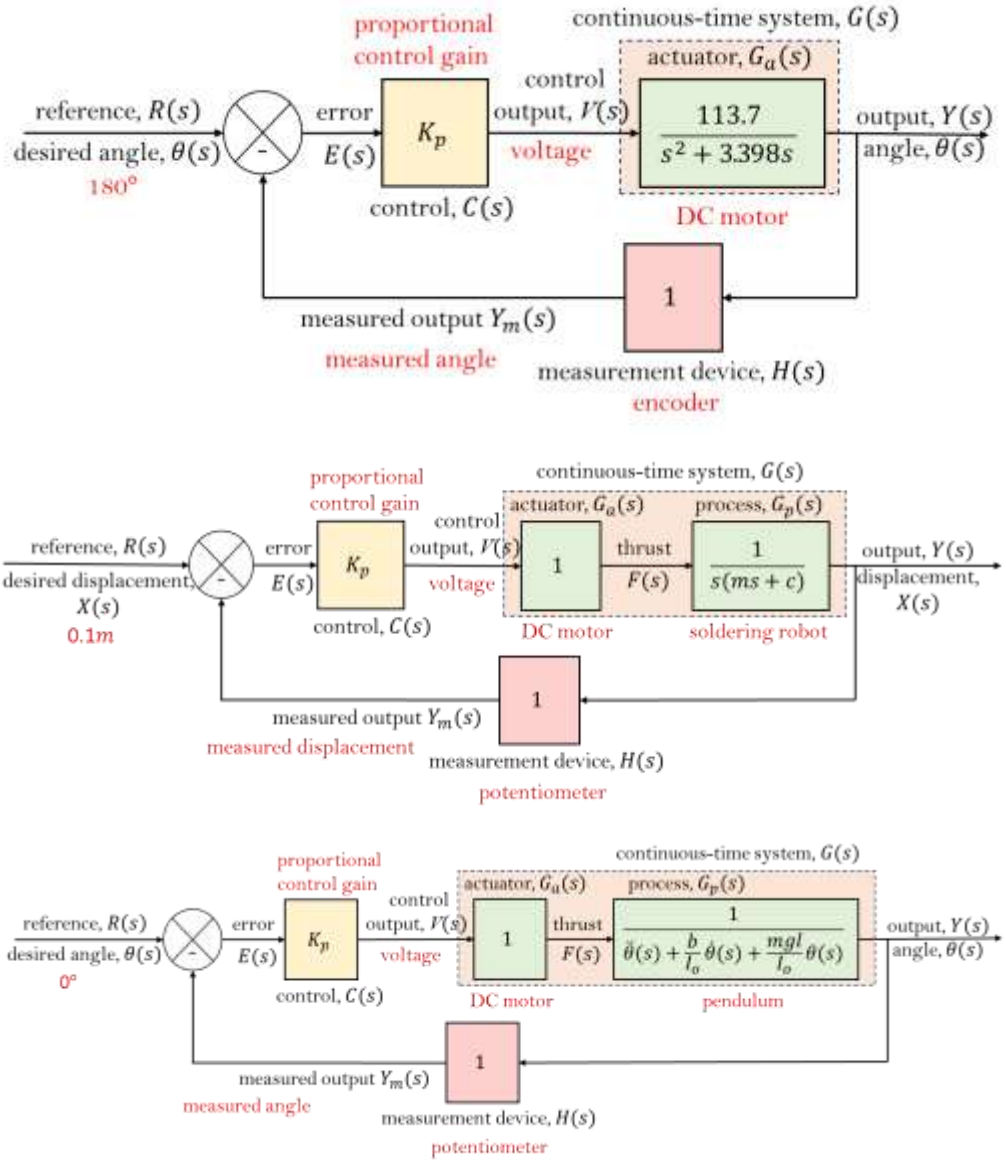


Figure 2: *Closed-Loop Control Systems of Engineering Problems: Water Tank (Top), DC Motor (Upper-Middle), Soldering Robot (Lower-Middle) and Pendulum (Bottom)*

4.0 Table of Transforms (Stage 9)

In the Laplace Transforms Learning Guide, the definition of the Laplace transform was used to transform time-domain functions into the complex frequency domain:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f\,(t)\,e^{-st}\,dt \tag{1}$$

The results for common functions were collected in a table of transforms, so that Equation (1) does not need to be evaluated each time. The same table can be read in reverse to obtain a time solution from a function of s. This is the inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \tag{2}$$

Table 2 extends the table of transforms with the functions most often needed in control engineering. Rows 2, 3, 7 and 8 contain the decaying exponential terms e^{-at} that appear in the response of most stable systems.

Table 2: *Table of Transforms Between the Time Domain and the Frequency Domain*

Row	Time domain, $f(t)$	Frequency domain, $F(s) = \mathcal{L}\{f(t)\}$
1	a (step of magnitude a)	$\frac{a}{s}$
2	e^{-at}	$\frac{1}{s+a}$
3	$t\,e^{-at}$	$\frac{1}{(s+a)^2}$
4	t^n	$\frac{n!}{s^{n+1}}$
5	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
6	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
7	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
8	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
9	$\sinh(at)$	$\frac{a}{s^2-a^2}$
10	$\cosh(at)$	$\frac{s}{s^2-a^2}$

Since the Laplace transform is linear, a constant multiplying a term can be taken outside the transform, and the transform of a sum is the sum of the transforms. For example, $\mathcal{L}^{-1}\{3F(s)\} = 3f(t)$. This property is used throughout the examples below.

Example 1: Step input

Determine the inverse Laplace transform of:

$$F(s) = \frac{3}{s}$$

Solution:

This is Row 1 of Table 2 with $a = 3$, a step of magnitude 3:

$$\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3$$

Example 2: First-order transfer function (general form)

Determine the inverse Laplace transform of the following first-order transfer function, where a is a constant:

$$G(s) = \frac{1}{s+a}$$

Solution:

This is Row 2 of Table 2:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

Note that the coefficient a has not been given a specific value in this example.

Example 3: First-order transfer function

Determine the inverse Laplace transform of:

$$G(s) = \frac{1}{s+4}$$

Solution:

This is Row 2 of Table 2 with $a = 4$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$$

Example 4: Second-order transfer function

Determine the inverse Laplace transform of:

$$G(s) = \frac{4}{s^2+4}$$

Solution:

The denominator matches Row 5 of Table 2 with $\omega^2 = 4$, so $\omega = 2$. Row 5 requires the numerator to be $\omega = 2$, so the numerator is written as 2×2 and the constant 2 is taken outside:

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{2\,\frac{2}{s^2+2^2}\right\} = 2\sin(2t)$$

5.0 Poles and the Quadratic Formula

The quadratic formula will already be familiar, but perhaps not yet in an engineering application. In this section, it is used to find the roots of the denominator of a second-order transfer function. In control engineering, the roots of the denominator of a transfer function are known as the **poles** of the system, and the roots of the numerator are known as the **zeros**. The poles and zeros are important because they determine the characteristics of the system response, as later Learning Guides will show. Knowing the type of poles also tells you which form of partial fractions, and which rows of the table of transforms, will be needed.

The quadratic formula is:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

where a, b and c are the coefficients of the denominator $as^2 + bs + c$. The general form of a second-order transfer function is:

$$G(s) = \frac{K\beta}{s^2 + \alpha s + \beta} \tag{4}$$

It is normal for the leading coefficient of the denominator of a transfer function to be normalised to 1, whatever the order of the system. Hence, for Equation (4), $a = 1$, $b = \alpha$ and $c = \beta$.

There are three possible types of poles. Which one applies is determined by the discriminant, $b^2 - 4ac$, as shown in the examples below.

Two equal real poles, where $b^2 - 4ac = 0$

Consider the following second-order transfer function:

$$G(s) = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)(s+2)}$$

The factorised form shows that the denominator has two equal real roots, $s_{1,2} = -2$. The discriminant confirms this: $4^2 - 4(1)(4) = 0$.

Two different real poles, where $b^2 - 4ac > 0$

Consider the following second-order transfer function:

$$G(s) = \frac{2}{s^2 + 3s + 2} = \frac{2}{(s+2)(s+1)}$$

The factorised form shows that the denominator has two different real roots, $s_1 = -2$ and $s_2 = -1$. The discriminant confirms this: $3^2 - 4(1)(2) = 1 > 0$.

Two complex poles, where $b^2 - 4ac < 0$

Consider the following second-order transfer function:

$$G(s) = \frac{2}{s^2 + 3s + 6}$$

This denominator cannot be factorised into real terms. The discriminant is $3^2 - 4(1)(6) = -15 < 0$, and so there are two complex poles. Using Equation (3), the poles are:

$$s_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4(1)(6)}}{2(1)} = \frac{-3 \pm j\sqrt{15}}{2} = -1.5 \pm 1.94j$$

These complex conjugate poles have a negative real part (-1.5) and an imaginary part ($\pm 1.94j$). Complex poles lead to the oscillatory terms in Rows 7 and 8 of Table 2.

6.0 Partial Fractions (Stage 8)

Partial fractions allow a complicated function of s to be expressed as a sum of simpler standard forms. Each of these simpler terms can then be found in the table of transforms.

Table 3 shows the partial fraction form for different types of roots. First-, second- and third-order denominators with real and/or complex roots are considered, with the factor in the denominator given in the third column and the corresponding partial fractions in the fourth column. The same forms apply to an open-loop system transfer function and to a closed-loop control system transfer function.

Table 3: *Partial Fractions for Different Types of Roots*

Order	Root type	Factor in denominator	Partial fractions
1st	One real root	$(s+a)$	$\frac{A}{s+a}$
2nd	Two different real roots	$(s+a)(s+b)$	$\frac{A}{s+a} + \frac{B}{s+b}$
2nd	Two repeated real roots	$(s+a)^2$	$\frac{A}{s+a} + \frac{B}{(s+a)^2}$
2nd	Two complex roots	$s^2 + bs + c$	$\frac{As+B}{s^2+bs+c}$
3rd	One real root and two complex roots	$(s+a)(s^2+bs+c)$	$\frac{A}{s+a} + \frac{Bs+C}{s^2+bs+c}$

As an example, the output of a closed-loop control system when a unit step input is applied is:

$$Y(s) = G_{CL}(s)\,R(s) = \frac{1}{s}\left[\frac{1}{s^2+3s+2}\right] = \frac{1}{s(s+2)(s+1)}$$

This expression is not in Table 2, and so the time solution cannot be obtained directly. The denominator has three different real roots (0, -2 and -1), and so, from Table 3, its partial fraction form is:

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

Each of the three terms is now in Table 2 (Rows 1 and 2). The three coefficients A, B and C must be determined, and two methods can be used for this:

- **Equating coefficients:** both sides are multiplied by the denominator, the brackets are expanded, and the coefficients of each power of s are equated to give simultaneous equations. This method works for every type of root and is demonstrated in Example 5.
- **The cover-up rule:** for a term $\frac{A}{s+a}$ with a distinct real root, the coefficient is found by multiplying $Y(s)$ by $(s+a)$ and then setting $s = -a$. This is quicker, but only applies directly to distinct real roots.

Applying the cover-up rule to the example above gives:

$$A = [s\,Y(s)]_{s=0} = \frac{1}{(0+2)(0+1)} = 0.5, \qquad B = [(s+2)\,Y(s)]_{s=-2} = \frac{1}{(-2)(-2+1)} = 0.5$$

$$C = [(s+1)\,Y(s)]_{s=-1} = \frac{1}{(-1)(-1+2)} = -1$$

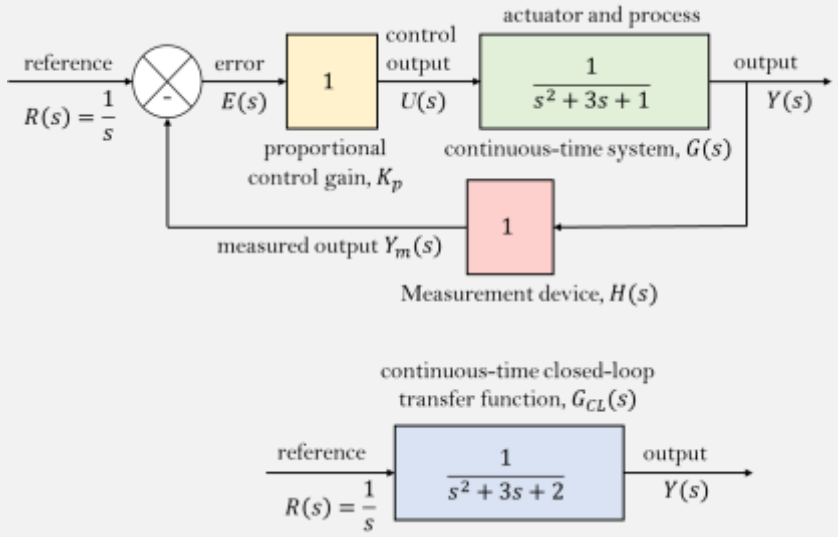
Once the coefficients are known, the table of transforms can be used (Stage 9) to give the time solution (Stage 10). The full process is shown in Section 7.0, where the same coefficients are obtained by equating coefficients.

7.0 Time Solution Using the Inverse Laplace Transform

In this section, a closed-loop control system is used as an illustrative example, and Stages 6 to 10 of Figure 1 are followed in turn.

Example 5: Time solution of a closed-loop control system

Consider the closed-loop control system shown below, which has a proportional control gain $K_p = 1$, an ideal sensor and a unit step reference.



Undertake the following tasks:

i. determine the output response as a function of time, that is, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

ii. sketch the system output and the reference on the same graph

Stage 6: block diagram algebra

With $N(s) = 1$, $D(s) = s^2 + 3s + 1$, $K_p = 1$ and $H(s) = 1$, the closed-loop transfer function from the Block Diagram Algebra Learning Guide is:

$$G_{CL}(s) = \frac{K_p\, N(s)}{D(s) + K_p\, H(s)\, N(s)} = \frac{1}{s^2 + 3s + 1 + 1} = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)}$$

Stage 7: applying the input

A unit step reference is modelled as $R(s) = 1/s$. The output is therefore:

$$Y(s) = G_{CL}(s)\, R(s) = \frac{1}{s}\,\frac{1}{(s+2)(s+1)} = \frac{1}{s(s+2)(s+1)}$$

Stage 8: partial fractions

As the table of transforms cannot be used directly, $Y(s)$ is broken down into partial fractions:

$$Y(s) = \frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

The coefficients are found by equating coefficients. Multiplying both sides by $s(s+2)(s+1)$ gives:

$$1 = A(s+2)(s+1) + Bs(s+1) + Cs(s+2)$$

Expanding the brackets gives:

$$1 = A(s^2 + 3s + 2) + B(s^2 + s) + C(s^2 + 2s)$$

Equating the coefficients of each power of s on both sides gives three simultaneous equations:

s^0: $1 = 2A$ s^1: $0 = 3A + B + 2C$ s^2: $0 = A + B + C$

From the s^0 equation, $\mathbf{A = 0.5}$. To eliminate C, the s^2 equation is multiplied by 2 and subtracted from the s^1 equation:

$$0 = (3A + B + 2C) - (2A + 2B + 2C) = A - B$$

Since $A = 0.5$, therefore $\mathbf{B = 0.5}$. Substituting A and B into the s^2 equation gives $0 = 0.5 + 0.5 + C$, and so $\mathbf{C = -1}$. The partial fractions are therefore:

$$Y(s) = \frac{0.5}{s} + \frac{0.5}{s+2} - \frac{1}{s+1}$$

These agree with the values found using the cover-up rule in Section 6.0.

Stage 9: table of transforms

Each term is now in Table 2, and so the inverse Laplace transform is applied term by term:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{0.5}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{0.5}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

Stage 10: time solution

Using Rows 1 and 2 of Table 2 gives the time solution:

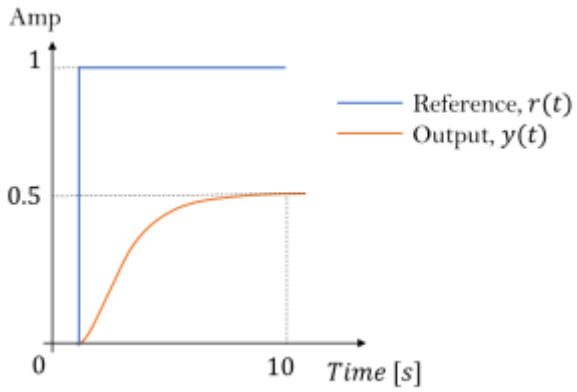
$$y(t) = 0.5 + 0.5e^{-2t} - e^{-t}$$

As $t \to \infty$, both exponential terms decay to zero, so the constant term (0.5) is the final (steady-state) value of the output.

To sketch the response, the time solution has been evaluated from 0 to 10 seconds in the table below. By 10 seconds, the output has reached its final value of 0.5, and beyond this the system is in steady state (there is no further change in the output).

Time, t [s]	Output, $y(t)$
0	0
1	0.1998
2	0.3738
3	0.4515
4	0.4819
5	0.4933
6	0.4975
7	0.4991
8	0.4997
9	0.4999
10	0.5000

The graph below has been sketched using the data in the table. The final value of the output is 0.5, whereas the reference has a value of 1.0. The closed-loop control system is therefore not performing as desired, because the reference has not been met. The difference between the reference and the final value of the output is known as the **steady-state error (SSE)**, which here is $1.0 - 0.5 = 0.5$.



8.0 Verifying the Solution Using MATLAB

The hand calculations for Example 5 can be checked using the MATLAB Editor:

i. MATLAB Script 1 determines the partial fractions and the time solution, by undertaking the inverse Laplace transform.

ii. MATLAB Script 2 plots the system response using the time solution from MATLAB Script 1.

MATLAB Script 1: determining the partial fractions and the time solution using the inverse Laplace transform

```
clear                       % clear workspace
clc                         % clear command window
close all                   % close open figures

%% Partial fractions
b = [1];                    % numerator of Y(s)
a = [1 3 2 0];              % denominator: s^3+3s^2+2s
[r, p, k] = residue(b, a)   % r: coefficients, p: poles

%% Inverse Laplace
syms s t                    % symbolic variables
Y = 1/(s*(s+2)*(s+1));      % closed-loop output, Y(s)
y = ilaplace(Y)             % time solution, y(t)
```

After clicking **Run**, the following should appear in the MATLAB Command Window. Each coefficient in `r` corresponds to the pole in the same position in `p`, so the partial fractions are $0.5/(s+2) - 1/(s+1) + 0.5/s$, as found by hand.

```
r =
    0.5000
```

```
   -1.0000
    0.5000

p =
    -2
    -1
     0

k =
     []

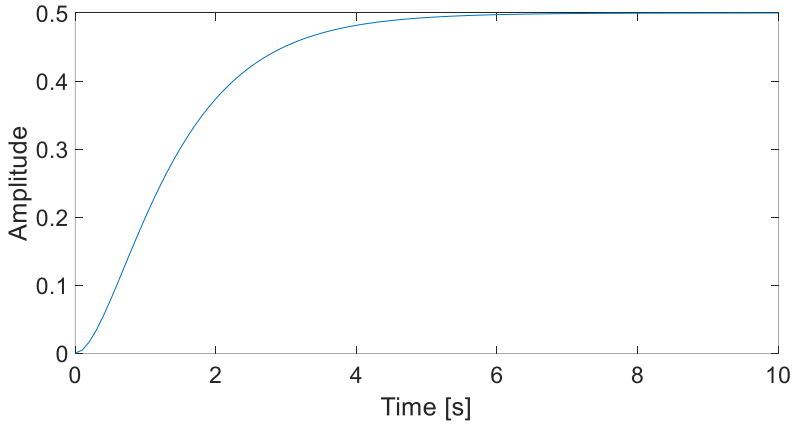
y =
exp(-2*t)/2 - exp(-t) + 1/2
```

MATLAB Script 2: plotting the system response

```
clear                               % clear workspace
clc                                 % clear command window
close all                           % close open figures

t = 0:0.1:10;                       % simulation time
y = exp(-2*t)/2 - exp(-t) + 1/2;    % time solution
plot(t, y)                          % graphical output
set(gca, 'FontSize', 30);           % text size 30
xlabel('Time [s]');                 % x-axis label
ylabel('Amplitude');                % y-axis label
```

After clicking **Run**, the following graph should appear:



The output response is the same as that obtained from the hand calculations. The unit step reference is not shown; to include it, add `hold on` and `plot(t, ones(size(t)), '--')` after the `plot(t, y)` command.

9.0 Exercises

Consider the following closed-loop transfer functions, where a unit step input is used to model the reference:

a) $G_{CL}(s) = \frac{12}{(s+3)(s+4)}$

b) $G_{CL}(s) = \frac{4}{(s+2)(s+2)}$

c) $G_{CL}(s) = \frac{2}{s^2+6s+8}$

d) $G_{CL}(s) = \frac{s^2+3s+2}{2s^2+6s+2}$

For each transfer function, undertake the following tasks:

i. determine the output response as a function of time, that is, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$
ii. sketch the system output response, including the unit step input (the reference)

Fully worked solutions to the exercises can be found in **Appendix A**. A step-by-step video solution for Question a can be found on YouTube. Note that in Appendix A, MATLAB has been used to plot the system output response for part ii, and the reference is not included.

Once you are comfortable with the above, determine the time solution for the closed-loop control systems in Figure 2. For these, a combination of hand calculations (Stages 6 and 7) and MATLAB (Stages 8 and 9) could be used to determine the time solution (Stage 10). Solutions are not provided for these systems, as they are used as examples in the Learning Guides that follow.

10.0 Summary

The time solution is important when analysing the performance of an open-loop system or a closed-loop control system. This Learning Guide showed how the time solution of a transfer function is determined when a known input is applied, using the inverse Laplace transform.

Ten stages for determining the time solution were set out. Stages 1 to 7 model the engineering problem and were covered in earlier Learning Guides, and this guide focused on the remaining stages. The table of transforms was extended, and the quadratic formula was used to identify the type of poles of a second-order transfer function. Partial fractions, found by equating coefficients or by the cover-up rule, were used to break the output into terms that appear in the table. A complete example was worked through by hand, and MATLAB scripts were provided to check the result. Further exercises were given that can be solved by hand and/or using MATLAB. Once you are confident with the hand calculations, you are strongly encouraged to use MATLAB to check your solutions.

The key progression introduced in this guide can be summarised as:

closed-loop transfer function → apply input → partial fractions → inverse Laplace transform → time solution

The time solution obtained here revealed a steady-state error. Later Learning Guides use these ideas to analyse system performance and to design controllers that meet the requirements.

Appendix A: Fully Worked Solutions to Exercises

In every question, the closed-loop output is $Y(s) = R(s)\,G_{CL}(s)$, where $R(s) = 1/s$ is the unit step reference.

Question a: $G_{CL}(s) = \frac{12}{(s+3)(s+4)}$

i. Applying the unit step input gives:

$$Y(s) = \frac{1}{s}\,\frac{12}{(s+3)(s+4)} = \frac{12}{s(s+3)(s+4)}$$

To use the table of transforms, partial fractions are used:

$$\frac{12}{s(s+3)(s+4)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}$$

Multiplying both sides by $s(s+3)(s+4)$ and expanding the brackets gives:

$$12 = A(s^2 + 7s + 12) + B(s^2 + 4s) + C(s^2 + 3s)$$

Equating the coefficients of each power of s gives:

$$s^0\colon\ 12 = 12A \qquad s^1\colon\ 0 = 7A + 4B + 3C \qquad s^2\colon\ 0 = A + B + C$$

From the s^0 equation, $A = 1$. Multiplying the s^2 equation by 3 and subtracting it from the s^1 equation eliminates C:

$$0 = 4A + B \quad \Rightarrow \quad B = -4$$

Substituting A and B into the s^2 equation gives $0 = 1 - 4 + C$, so $C = 3$. Hence:

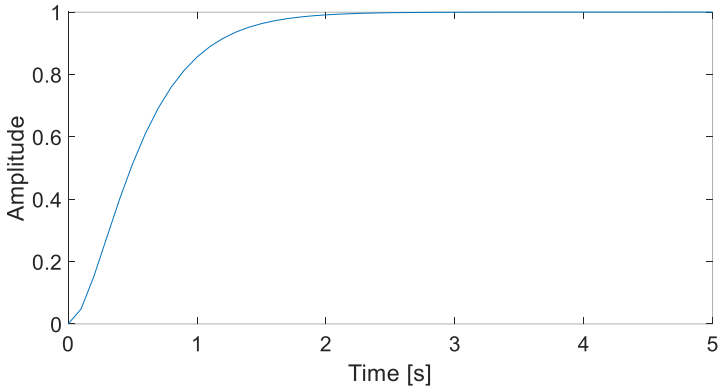
$$Y(s) = \frac{1}{s} - \frac{4}{s+3} + \frac{3}{s+4}$$

Using Rows 1 and 2 of the table of transforms gives the time solution:

$$y(t) = 1 - 4e^{-3t} + 3e^{-4t}$$

ii. A MATLAB script similar to the following could be used:

```
t = 0:0.1:5;                          % simulation time
y = 1 - 4*exp(-3*t) + 3*exp(-4*t);    % time solution
plot(t, y)                            % graphical output
set(gca, 'FontSize', 30);             % text size 30
xlabel('Time [s]');                   % x-axis label
ylabel('Amplitude');                  % y-axis label
```



Question b: $G_{CL}(s) = \frac{4}{(s+2)(s+2)}$

i. Applying the unit step input gives:

$$Y(s) = \frac{1}{s}\frac{4}{(s+2)(s+2)} = \frac{4}{s(s+2)^2}$$

The denominator has a repeated root at $s = -2$, so from Table 3 the partial fractions are:

$$\frac{4}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Multiplying both sides by $s(s+2)^2$ and expanding the brackets gives:

$$4 = A(s^2 + 4s + 4) + B(s^2 + 2s) + Cs$$

Equating the coefficients of each power of s gives:

$$s^0\text{: } 4 = 4A \qquad s^1\text{: } 0 = 4A + 2B + C \qquad s^2\text{: } 0 = A + B$$

From the s^0 equation, $A = 1$, and from the s^2 equation, $B = -A = -1$. Substituting into the s^1 equation gives $0 = 4 - 2 + C$, so $C = -2$. Hence:

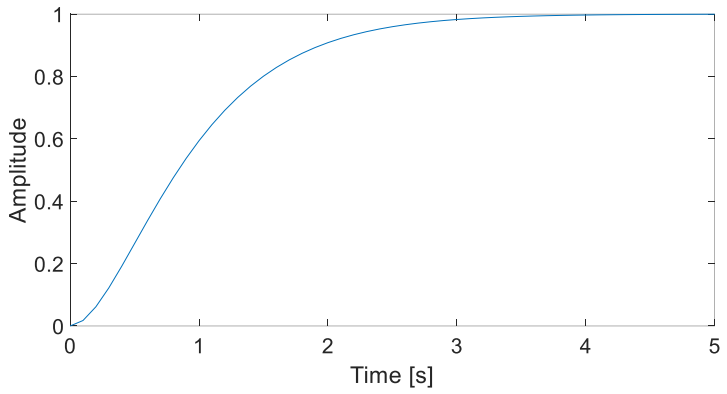
$$Y(s) = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}$$

Using Rows 1, 2 and 3 of the table of transforms gives the time solution:

$$y(t) = 1 - e^{-2t} - 2t\,e^{-2t}$$

ii. A MATLAB script similar to the following could be used:

```
t = 0:0.1:5;                          % simulation time
y = 1 - exp(-2*t) - 2*t.*exp(-2*t);   % time solution
plot(t, y)                            % graphical output
set(gca, 'FontSize', 30);             % text size 30
xlabel('Time [s]');                   % x-axis label
ylabel('Amplitude');                  % y-axis label
```



Question c: $G_{CL}(s) = \frac{2}{s^2+6s+8}$

i. The discriminant of the denominator is $6^2 - 4(1)(8) = 4 > 0$, so there are two different real poles. Using the quadratic formula, $s_{1,2} = (-6 \pm 2)/2 = -2$ and -4, so $s^2 + 6s + 8 = (s+2)(s+4)$. Applying the unit step input gives:

$$Y(s) = \frac{1}{s}\frac{2}{s^2+6s+8} = \frac{2}{s(s+2)(s+4)}$$

To use the table of transforms, partial fractions are used:

$$\frac{2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

Multiplying both sides by $s(s+2)(s+4)$ and expanding the brackets gives:

$$2 = A(s^2+6s+8) + B(s^2+4s) + C(s^2+2s)$$

Equating the coefficients of each power of s gives:

$$s^0\colon\ 2 = 8A \qquad s^1\colon\ 0 = 6A + 4B + 2C \qquad s^2\colon\ 0 = A + B + C$$

From the s^0 equation, $A = 0.25$. Multiplying the s^2 equation by 2 and subtracting it from the s^1 equation eliminates C:

$$0 = 4A + 2B \quad \Rightarrow \quad B = -0.5$$

Substituting A and B into the s^2 equation gives $0 = 0.25 - 0.5 + C$, so $C = 0.25$. Hence:

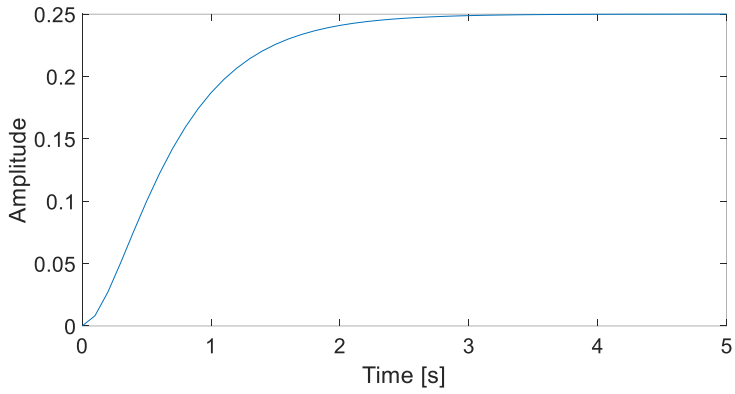
$$Y(s) = \frac{0.25}{s} - \frac{0.5}{s+2} + \frac{0.25}{s+4}$$

Using Rows 1 and 2 of the table of transforms gives the time solution:

$$y(t) = 0.25 - 0.5e^{-2t} + 0.25e^{-4t}$$

ii. A MATLAB script similar to the following could be used:

```
t = 0:0.1:5;                                % simulation time
y = 0.25 - 0.5*exp(-2*t) + 0.25*exp(-4*t);  % time solution
plot(t, y)                                  % graphical output
set(gca, 'FontSize', 30);                   % text size 30
xlabel('Time [s]');                         % x-axis label
ylabel('Amplitude');                        % y-axis label
```



Question d: $G_{CL}(s) = \frac{s^2+3s+2}{2s^2+6s+2}$

i. First, the denominator is normalised by taking out the factor of 2:

$$G_{CL}(s) = \frac{s^2 + 3s + 2}{2(s^2 + 3s + 1)}$$

The discriminant of $s^2 + 3s + 1$ is $3^2 - 4(1)(1) = 5 > 0$, so there are two different real poles:

$$s_{1,2} = \frac{-3 \pm \sqrt{5}}{2} \quad \Rightarrow \quad s_1 = -0.382, \quad s_2 = -2.618$$

So $s^2 + 3s + 1 = (s + 0.382)(s + 2.618)$. Applying the unit step input, and factorising the numerator as $(s + 1)(s + 2)$, gives:

$$Y(s) = \frac{1}{s}\frac{(s+1)(s+2)}{2(s+0.382)(s+2.618)} = \frac{(s+1)(s+2)}{2s(s+0.382)(s+2.618)}$$

All three roots of the denominator are distinct and real, so the partial fractions are:

$$Y(s) = \frac{A}{s} + \frac{B}{s+0.382} + \frac{C}{s+2.618}$$

The cover-up rule is the quickest method here:

$$A = [s\,Y(s)]_{s=0} = \frac{(1)(2)}{2(0.382)(2.618)} = \frac{2}{2(1.000)} = 1$$

$$B = [(s+0.382)\,Y(s)]_{s=-0.382} = \frac{(0.618)(1.618)}{2(-0.382)(2.236)} = \frac{1.000}{-1.708} = -0.585$$

$$C = [(s+2.618)\,Y(s)]_{s=-2.618} = \frac{(-1.618)(-0.618)}{2(-2.618)(-2.236)} = \frac{1.000}{11.708} = 0.085$$

As a check, $A + B + C = 0.5$, which equals the coefficient of s^2 in the numerator of $Y(s)$ divided by 2, as it should. Hence:

$$Y(s) = \frac{1}{s} - \frac{0.585}{s+0.382} + \frac{0.085}{s+2.618}$$

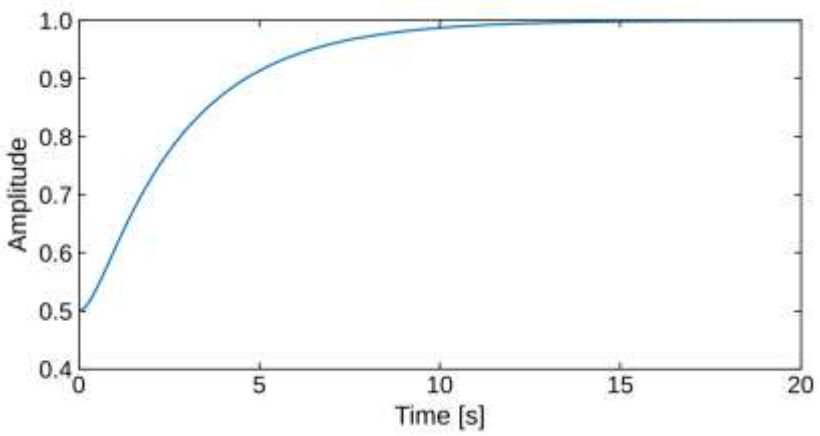
Using Rows 1 and 2 of the table of transforms gives the time solution:

$$y(t) = 1 - 0.585e^{-0.382t} + 0.085e^{-2.618t}$$

In exact form, $y(t) = 1 - \frac{3\sqrt{5}+5}{20}\, e^{-\frac{3-\sqrt{5}}{2}t} + \frac{3\sqrt{5}-5}{20}\, e^{-\frac{3+\sqrt{5}}{2}t}$. Note that $y(0) = 0.5$: because the numerator and denominator of $G_{CL}(s)$ have the same order, part of the step passes straight through to the output. The output then settles to its final value of 1.

ii. A MATLAB script similar to the following could be used:

```
t = 0:0.1:20;              % simulation time
y = 1 - ((3*sqrt(5)+5)/20)*exp(-0.5*(3-sqrt(5))*t) ...
      + ((3*sqrt(5)-5)/20)*exp(-0.5*(3+sqrt(5))*t);
plot(t, y)                 % graphical output
set(gca, 'FontSize', 30);  % text size 30
xlabel('Time [s]');        % x-axis label
ylabel('Amplitude');       % y-axis label
```



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