Block Diagram Algebra

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1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the Block Diagram Algebra Learning Guide*

ID	Learning Outcome	Achieved
1.	Identify the components and signals of open-loop and closed-loop control systems and represent them in block diagram form.	☐
2.	Apply block diagram algebra to derive the transfer function of an open-loop system, with and without a disturbance.	☐
3.	Apply block diagram algebra to derive the transfer function of a closed-loop control system, with and without a disturbance, and explain how feedback reduces the effect of a disturbance.	☐
4.	Derive the closed-loop transfer function of a control system containing a proportional control gain, an ideal sensor and an ideal actuator.	☐

2.0 Introduction

In the Laplace Transforms Learning Guide, the Laplace transform was applied to linear time-invariant (LTI) ordinary differential equations (ODEs), and the transformed equations were rearranged into transfer functions. The Laplace transforms of the step and ramp inputs were also derived.

A closed-loop control system is made up of several components connected together, as shown in Figure 1. Each block in the figure, and the reference signal itself, can be modelled mathematically in the Laplace domain:

- Reference, denoted $R(s)$
- Control (the controller), denoted $C(s)$
- Actuator, denoted $G_a(s)$
- Process, denoted $G_p(s)$
- Measurement device (the sensor), denoted $H(s)$

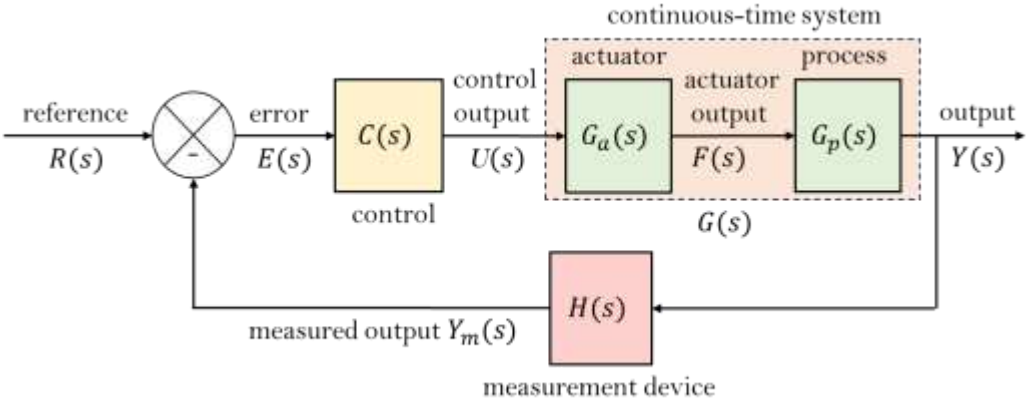


Figure 1: *Closed-Loop Control System (Continuous-Time) Schematic used in Control Engineering for Modelling, Simulation and Model-Based Control System Design*

An important question therefore arises:

How can the transfer functions of the individual components be combined into a single model of the complete system?

Because the Laplace domain allows transfer functions to be manipulated using the ordinary rules of algebra, the individual blocks can be combined into a single closed-loop transfer function. This process is known as block diagram algebra. It rearranges the terms of the model (for example, $R(s)$, $C(s)$, $G_a(s)$, $G_p(s)$ and $H(s)$) into the form:

$$G_{CL}(s) = \frac{Y(s)}{R(s)}$$

where $G_{CL}(s)$ denotes the closed-loop transfer function, $Y(s)$ denotes the output and $R(s)$ denotes the reference. This should not be confused with the open-loop transfer function, $G(s) = Y(s)/U(s)$, which relates the output to the control input rather than to the reference. The closed-loop transfer function is used in the analysis and design stages of the model-based design process.

This Learning Guide first introduces the elements of a block diagram and the basic rules for combining blocks. The open-loop transfer function is then derived, both with and without a disturbance, followed by the closed-loop transfer function, again with and without a disturbance. Finally, the controller is replaced by a proportional control gain, K_p, and the closed-loop transfer function of a DC motor angle control system is derived.

Why is Block Diagram Algebra Important?

Block diagram algebra is important for both open-loop systems and closed-loop control systems because:

i. Inspecting the transfer function (open-loop or closed-loop) provides information on the transient and steady-state parts of the system response. These are much harder to determine in the time domain, as later Learning Guides will show.
ii. The closed-loop transfer function allows the system performance to be predicted and compared with the design requirements, for example the overshoot and settling time of the response.
iii. The controller can be configured and tuned using the form of the closed-loop transfer function, both to achieve the required performance and to reject disturbances.

These points will become clearer as the closed-loop transfer function is used in later Learning Guides.

3.0 Block Diagram Elements and Rules

As introduced above, the purpose of block diagram algebra is to capture a mathematical model of a system, which is then used to analyse an open-loop system or to design a closed-loop control system. Referring to Figure 1, a block diagram is made up of three elements:

i. **Blocks**, which represent components and contain their transfer functions.
ii. **Lines with arrows**, which represent signals and the direction in which they flow.
iii. **Summing junctions**, which add or subtract signals. The sign at each input of the junction shows whether that signal is added or subtracted.

The output of a block is always its input multiplied by its transfer function. For example, for the actuator block, $F(s) = G_a(s)U(s)$. It is assumed that connecting the output of one block to the input of another does not change the behaviour of the first block (that is, there are no loading effects).

Using these elements, a small number of rules allow any block diagram to be reduced to a single equivalent transfer function. The three most common rules are summarised in Table 2. The feedback rule is derived in full in Section 5.0.

Table 2: *Basic Rules of Block Diagram Algebra*

Configuration	Description	Equivalent transfer function
Series (cascade)	The output of $G_1(s)$ is the input to $G_2(s)$	$G_1(s)\,G_2(s)$
Parallel	The same input is applied to $G_1(s)$ and $G_2(s)$ and their outputs are added or subtracted	$G_1(s) \pm G_2(s)$
Negative feedback	$G(s)$ in the forward path and $H(s)$ in the feedback path, with the feedback subtracted	$\frac{G(s)}{1 + G(s)H(s)}$

4.0 Open-Loop Systems

Open-Loop Transfer Function

The open-loop system schematic used in control engineering is shown in Figure 2. The control output $U(s)$ drives the actuator, which produces the actuator output $F(s)$, and this drives the process to give the output $Y(s)$. There is no measurement of the output and no feedback.

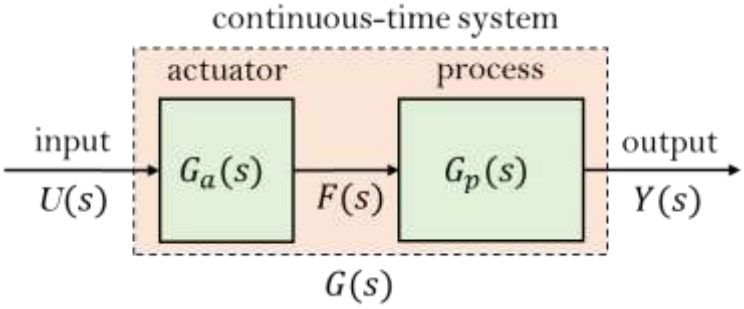


Figure 2: *Open-Loop System (Continuous-Time) Schematic used in Control Engineering for Modelling, Simulation and Model-Based Design*

The output of each block is its input multiplied by its transfer function:

$$F(s) = G_a(s)\,U(s) \tag{1}$$

$$Y(s) = G_p(s)\,F(s) \tag{2}$$

Equations (1) and (2) describe how the blocks and signals are connected. Substituting Equation (1) into Equation (2) gives:

$$Y(s) = G(s)\,U(s) \tag{3}$$

where $G(s) = G_a(s)\,G_p(s)$ is the open-loop transfer function. This is an example of the series rule in Table 2: the actuator and process are in series, so their transfer functions multiply.

Open-Loop System Experiencing a Disturbance

Almost all engineering systems experience a disturbance of some type, for example a gust of wind acting on a vehicle or a change in the load on a motor. When the open-loop system experiences a disturbance, $D(s)$, it can be represented as shown in Figure 3, where the disturbance is added to (or subtracted from) the actuator output before it reaches the process.

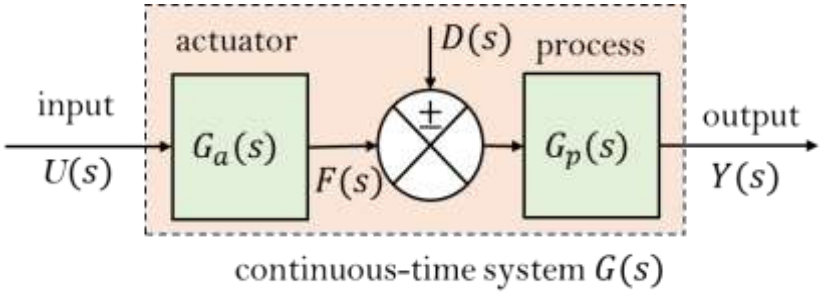


Figure 3: *Open-Loop System Experiencing a Disturbance (Continuous-Time) Schematic used in Control Engineering for Modelling, Simulation and Model-Based Design*

The actuator output is still given by Equation (1), $F(s) = G_a(s)U(s)$. The output of the process is now:

$$Y(s) = G_p(s)[F(s) \pm D(s)] \tag{4}$$

Substituting Equation (1) into Equation (4) gives the output of an open-loop system experiencing a disturbance:

$$Y(s) = G_p(s)[G_a(s)\,U(s) \pm D(s)] = G(s)\,U(s) \pm G_p(s)\,D(s) \tag{5}$$

Equation (5) shows that the effect of the disturbance, $G_p(s)D(s)$, passes directly to the output. Because the output is not measured, an open-loop system has no way of detecting or correcting this effect. This is one of the main motivations for using feedback.

5.0 Closed-Loop Control Systems

Closed-Loop Transfer Function

The closed-loop control system schematic used in control engineering was shown in Figure 1. The output is measured by the measurement device and compared with the reference at the summing junction, and the resulting error drives the controller.

The control output is the controller transfer function multiplied by the error:

$$U(s) = C(s)\,E(s) \tag{6}$$

From Equation (3), the output of the system is $Y(s) = G(s)U(s)$, where $G(s) = G_a(s)G_p(s)$. Substituting Equation (6) gives the relationship between the error and the output:

$$Y(s) = G(s)\,C(s)\,E(s) \tag{7}$$

The error is the difference between the reference and the measured output:

$$E(s) = R(s) - H(s)\,Y(s) = R(s) - Y_m(s) \tag{8}$$

where $Y_m(s) = H(s)Y(s)$ is the measured output. Substituting Equation (8) into Equation (7) gives:

$$Y(s) = G(s)\,C(s)[R(s) - H(s)\,Y(s)] \tag{9}$$

Expanding Equation (9) gives:

$$Y(s) = G(s)\,C(s)\,R(s) - H(s)\,G(s)\,C(s)\,Y(s) \tag{10}$$

Collecting the $Y(s)$ terms on the left-hand side of Equation (10) gives:

$$Y(s)[1 + H(s)\,G(s)\,C(s)] = G(s)\,C(s)\,R(s) \tag{11}$$

Dividing Equation (11) throughout by $[1 + H(s)G(s)C(s)]$ gives:

$$Y(s) = \left[\frac{G(s)\,C(s)}{1 + H(s)\,G(s)\,C(s)}\right] R(s) = G_{CL}(s)\,R(s) \tag{12}$$

or, equivalently, the closed-loop transfer function is:

$$G_{CL}(s) = \frac{Y(s)}{R(s)} = \frac{G(s)\,C(s)}{1 + H(s)\,G(s)\,C(s)} \tag{13}$$

Equation (13) is the negative feedback rule from Table 2, with $G(s)C(s)$ in the forward path and $H(s)$ in the feedback path. A useful way to remember it is:

$$G_{CL}(s) = \frac{\text{forward path}}{1 + \text{loop}}$$

where the “loop” is the product of every transfer function around the feedback loop. The denominator, $1 + H(s)G(s)C(s)$, is particularly important: setting it equal to zero gives the characteristic equation, which determines the stability and dynamic behaviour of the closed-loop system, as later Learning Guides will show.

Equation (13) is used extensively in the design and analysis of closed-loop control systems, where the aim is to achieve a desired reference with a desired dynamic performance. For example, a vehicle cruise control system has a reference (the desired vehicle speed), and the controller is designed so that this speed is reached within a specified time and with a smooth response.

Closed-Loop System Experiencing a Disturbance

A closed-loop control system experiencing a disturbance is shown in Figure 4. The aim of the closed-loop control system is to achieve the reference even when disturbances act on the system.

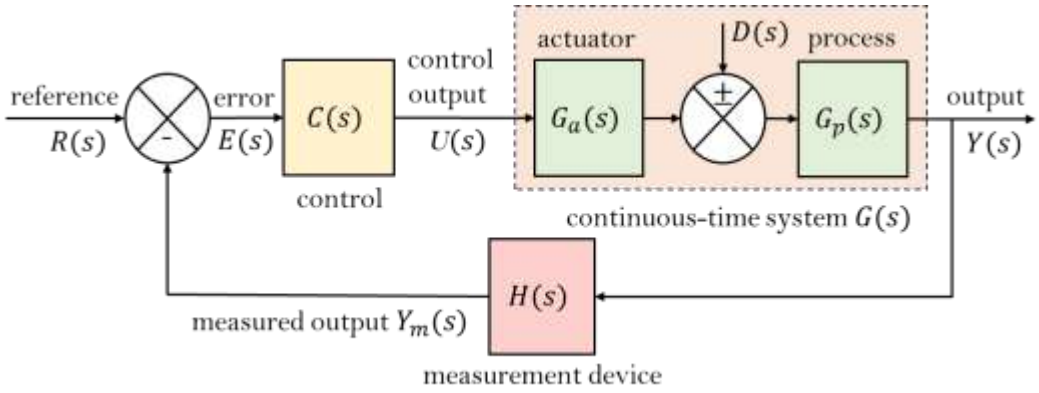


Figure 4: *Closed-Loop Control System Experiencing a Disturbance (Continuous-Time) Schematic used in Control Engineering for Modelling, Simulation and Model-Based Design*

The control output is still given by Equation (6), $U(s) = C(s)E(s)$. From Equation (5), the output of the system with the disturbance is:

$$Y(s) = G_p(s)[G_a(s)\,U(s) \pm D(s)] \tag{14}$$

Substituting Equation (6) into Equation (14) gives the relationship between the error, the disturbance and the output:

$$Y(s) = G_p(s)[G_a(s)\,C(s)\,E(s) \pm D(s)] \tag{15}$$

The error is given by Equation (8), $E(s) = R(s) - H(s)Y(s)$. Substituting this into Equation (15) gives:

$$Y(s) = G_p(s)[G_a(s)\,C(s)[R(s) - H(s)\,Y(s)] \pm D(s)] \tag{16}$$

Expanding Equation (16) gives:

$$Y(s) = G_a(s)\,C(s)\,G_p(s)\,R(s) - H(s)\,G_a(s)\,C(s)\,G_p(s)\,Y(s) \pm G_p(s)\,D(s) \tag{17}$$

Collecting the $Y(s)$ terms on the left-hand side gives:

$$Y(s)\big[1 + H(s)\,G_a(s)\,C(s)\,G_p(s)\big] = G_a(s)\,C(s)\,G_p(s)\,R(s) \pm G_p(s)\,D(s) \tag{18}$$

Dividing Equation (18) throughout by $\big[1 + H(s)G_a(s)C(s)G_p(s)\big]$ gives:

$$Y(s) = \left[\frac{G_a(s)\,C(s)\,G_p(s)}{1 + H(s)\,G_a(s)\,C(s)\,G_p(s)}\right]R(s) \pm \left[\frac{G_p(s)}{1 + H(s)\,G_a(s)\,C(s)\,G_p(s)}\right]D(s) \tag{19}$$

Equation (19) shows that the output is the sum of two parts: the response to the reference and the response to the disturbance. Both parts share the same denominator. In the case where there is no disturbance, $D(s) = 0$, and so:

$$G_{CL}(s) = \frac{Y(s)}{R(s)} = \frac{G_a(s)\,C(s)\,G_p(s)}{1 + H(s)\,G_a(s)\,C(s)\,G_p(s)} \tag{20}$$

Since $G(s) = G_a(s)G_p(s)$, Equation (20) is identical to Equation (13).

Comparing Equation (19) with Equation (5) shows the benefit of feedback. In the open-loop system, the disturbance reaches the output as $G_p(s)D(s)$. In the closed-loop system, the same term is divided by $1 + H(s)G_a(s)C(s)G_p(s)$, and so the larger the gain around the loop, the smaller the effect of the disturbance on the output. When modelling and simulating a system, it is therefore important to understand the disturbances it might experience, so that the controller can be configured and tuned to compensate for them. For the cruise control example, a typical disturbance is a head wind or a hill.

6.0 Proportional Control

Proportional Control Gain

Considering Figure 4 (ignoring the disturbance for now), the controller $C(s)$ is now replaced with a proportional control gain, K_p, as shown in Figure 5. Substituting $C(s) = K_p$ into Equation (13) gives:

$$G_{CL}(s) = \frac{Y(s)}{R(s)} = \frac{K_p\,G(s)}{1 + K_p\,H(s)\,G(s)} \tag{21}$$

The proportional control gain K_p is a constant that multiplies the error between the reference and the measured output to give the control output. It therefore applies a corrective action that is in direct proportion to the error. The value chosen for K_p changes the performance of the system: a higher K_p generally results in a faster response, but may cause overshoot and oscillation. Many control systems used in industry consist of just a proportional control gain. How K_p affects the system performance is studied using the final value theorem in a later Learning Guide, and explored further in the Learning Guides that follow it.

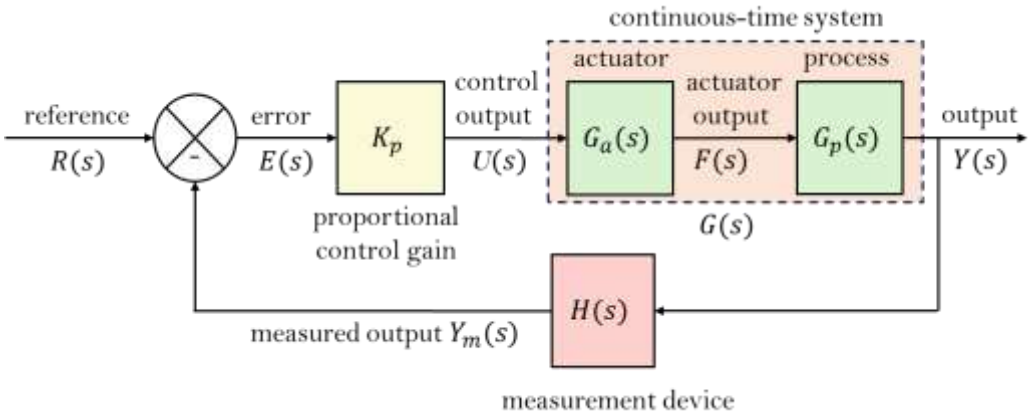


Figure 5: *Closed-Loop Control System Containing a Proportional Control Gain, K_p*

A Convenient Mathematical Form

When calculations are undertaken using Equation (21), it is convenient to write the open-loop transfer function as a numerator $N(s)$ divided by a denominator $D(s)$, that is, $G(s) = N(s)/D(s)$. The output is then:

$$Y(s) = R(s)\left[\frac{K_p \dfrac{N(s)}{D(s)}}{1 + K_p\, H(s)\, \dfrac{N(s)}{D(s)}}\right] \tag{22}$$

Multiplying the numerator and denominator by $D(s)$ gives the following form:

$$Y(s) = R(s)\left[\frac{K_p\, N(s)}{D(s) + K_p\, H(s)\, N(s)}\right] \tag{23}$$

Using Equation (23), only the numerator and denominator of the open-loop transfer function need to be substituted, which simplifies the working. The closed-loop transfer function is therefore:

$$G_{CL}(s) = \frac{Y(s)}{R(s)} = \frac{K_p\, N(s)}{D(s) + K_p\, H(s)\, N(s)} \tag{24}$$

Note that $D(s)$ here denotes the denominator of $G(s)$ and should not be confused with the disturbance in Section 5.0.

Ideal Sensor and Ideal Actuator

For simplicity, some components of a closed-loop control system are often modelled with constant values, as shown in Figure 6. The following assumptions are commonly applied:

- An **ideal sensor** is assumed, that is, $H(s) = 1$. The measured output is then equal to the actual output.
- When the process is the focus of the design, it is also convenient to assume an **ideal actuator**, that is, $G_a(s) = 1$. This assumes that the

actuator has no dynamics (no lag). This assumption is not made when the actuator itself is being controlled, as in Example 2.

With both assumptions, $G(s) = G_p(s) = N(s)/D(s)$ and Equation (24) becomes $G_{CL}(s) = K_p N(s)/\big[D(s) + K_p N(s)\big]$.

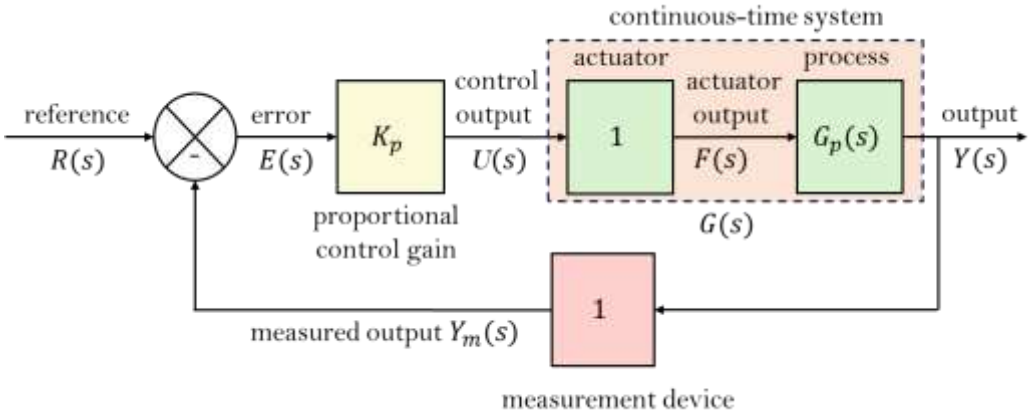


Figure 6: *Closed-Loop Control System Containing a Proportional Control Gain, K_p, an Ideal Sensor and an Ideal Actuator*

Example 1: Closed-loop transfer function of a first-order process

The first-order process from Example 3 of the Laplace Transforms Learning Guide has the transfer function:

$$G_p(s) = \frac{1}{s+2}$$

Assuming an ideal sensor, an ideal actuator and a proportional control gain of $K_p = 5$, determine the closed-loop transfer function.

Solution:

With $G_a(s) = 1$, the open-loop transfer function is $G(s) = G_p(s)$, so $N(s) = 1$ and $D(s) = s + 2$. With $H(s) = 1$, Equation (24) gives:

$$G_{CL}(s) = \frac{K_p\, N(s)}{D(s) + K_p\, N(s)} = \frac{5(1)}{(s+2) + 5(1)} = \frac{5}{s+7}$$

The closed-loop transfer function is still first order, but its denominator has changed from $s + 2$ to $s + 7$. The controller has therefore changed the dynamics of the system, as will be explored in later Learning Guides.

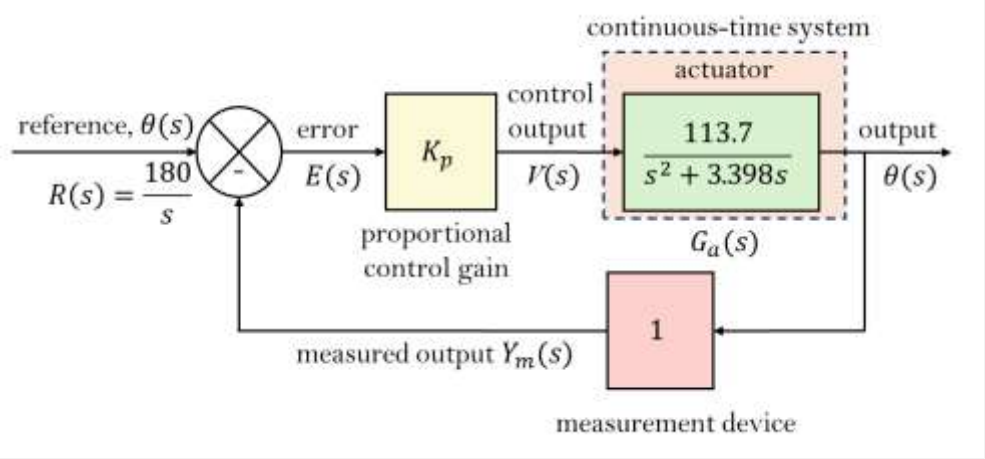
Example 2: Closed-loop DC motor angle control

In the ODE and Time Solution Learning Guide, the DC motor actuator was described by the ODE:

$$0.008795\,\ddot{\theta}(t) + 0.029889\,\dot{\theta}(t) = V_{in}(t)$$

where $\theta(t)$ is the motor angle and $V_{in}(t)$ is the input voltage. Feedback is introduced to create a closed-loop control system, with a proportional

control gain K_p in the forward path and an ideal sensor ($H(s) = 1$) in the feedback path, as shown in the figure below. The requirement is to achieve a motor angle of 180°.



Determine the transfer function of the DC motor, the output of the closed-loop control system and the closed-loop transfer function.

Solution:

DC motor transfer function: dividing the ODE throughout by 0.008795, applying the Laplace transform with zero initial conditions and rearranging, as in the Laplace Transforms Learning Guide, gives:

$$G_a(s) = \frac{\theta(s)}{V_{in}(s)} = \frac{113.7}{s^2 + 3.398s}$$

Reference: a constant angle of 180° is a step of magnitude 180, so from the table of transforms, $R(s) = 180/s$.

Output: the DC motor is the only block in the forward path, so $N(s) = 113.7$ and $D(s) = s^2 + 3.398s$. With $H(s) = 1$, Equation (23) gives:

$$\theta(s) = \frac{180}{s}\left[\frac{113.7K_p}{(s^2 + 3.398s) + 113.7K_p}\right] = \frac{180}{s}\left[\frac{113.7K_p}{s^2 + 3.398s + 113.7K_p}\right]$$

Closed-loop transfer function:

$$G_{CL}(s) = \frac{\theta(s)}{R(s)} = \frac{113.7K_p}{s^2 + 3.398s + 113.7K_p}$$

For example, with $K_p = 1$, the closed-loop transfer function becomes $G_{CL}(s) = 113.7/(s^2 + 3.398s + 113.7)$.

As later Learning Guides will show, the closed-loop transfer function is of interest because it defines how the system responds over time to its inputs, including disturbances. It determines the stability, transient response and steady-state behaviour of the system. This allows engineers to predict time-domain performance, optimise the controller design, and ensure that the system meets its requirements, such as the speed and accuracy of the response.

7.0 Exercises

Consider the open-loop systems shown in Figures 7 and 8. For each system, determine the closed-loop transfer function, $G_{CL}(s)$, of a closed-loop control system containing:

i. an ideal sensor
ii. a proportional control gain, K_p
iii. an ideal actuator

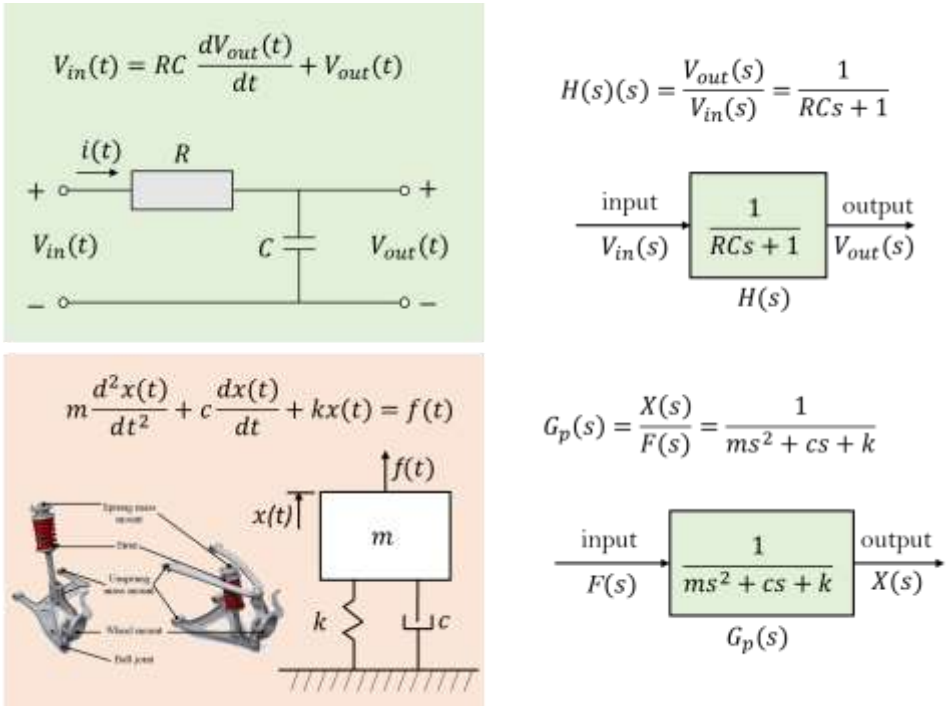


Figure 7: *Open-Loop Systems for the Exercises: RC Circuit (Top) and Suspension System (Bottom)*

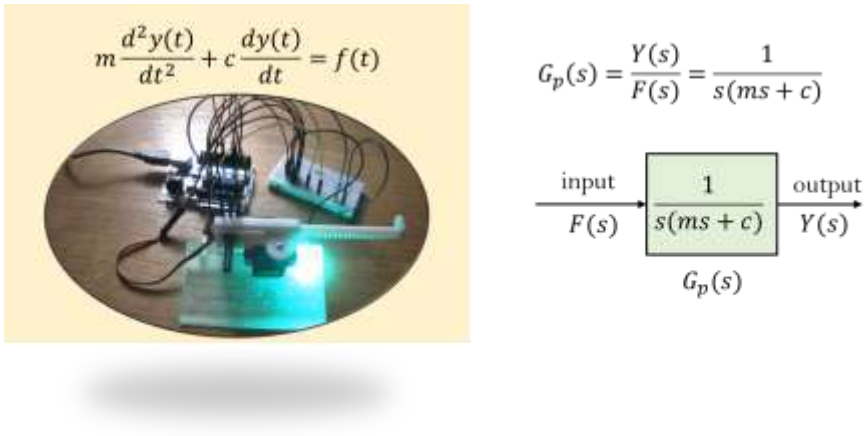


Figure 8: *Open-Loop System for the Exercises: Soldering Robot*

Answers to the above exercises can be found in **Appendix A**.

8.0 Summary

This Learning Guide showed how block diagram algebra is used to combine the transfer functions of individual components into a single model of a complete system. The elements of a block diagram (blocks, signal lines and summing junctions) were introduced, together with the series, parallel and negative feedback rules.

Block diagram algebra was then used to derive:

i. the open-loop transfer function
ii. the open-loop transfer function with a disturbance
iii. the closed-loop transfer function
iv. the closed-loop transfer function with a disturbance
v. the closed-loop transfer function with a proportional control gain, an ideal sensor and an ideal actuator

Comparing the open-loop and closed-loop results showed that feedback reduces the effect of a disturbance on the output. The closed-loop transfer function of a DC motor angle control system was also derived. The key progression introduced in this guide can be summarised as:

transfer functions → block diagram → closed-loop transfer function

The closed-loop transfer function is used throughout the analysis and design stages of later Learning Guides. In the Inverse Laplace Transform Learning Guide, it is used to determine the time solution of a system when a known input is applied.

Appendix A: Answers to Exercises

With an ideal sensor ($H(s) = 1$) and an ideal actuator ($G_a(s) = 1$), the open-loop transfer function is $G(s) = G_p(s) = N(s)/D(s)$, and Equation (24) gives $G_{CL}(s) = K_p N(s)/\big[D(s) + K_p N(s)\big]$.

RC circuit

The open-loop transfer function is $\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs+1}$ (labelled $H(s)$ in Figure 7, not to be confused with the measurement device), so $N(s) = 1$ and $D(s) = RCs + 1$. Therefore:

$$G_{CL}(s) = \frac{K_p}{RCs + 1 + K_p}$$

Suspension system (mass-spring-damper)

The open-loop transfer function is $G_p(s) = \frac{1}{ms^2+cs+k}$, so $N(s) = 1$ and $D(s) =$

$ms^2 + cs + k$. Therefore:

$$G_{CL}(s) = \frac{K_p}{ms^2 + cs + k + K_p}$$

Soldering robot

The open-loop transfer function is $G_p(s) = \frac{1}{s(ms+c)} = \frac{1}{ms^2+cs}$, so $N(s) = 1$ and $D(s) = ms^2 + cs$. Therefore:

$$G_{CL}(s) = \frac{K_p}{ms^2 + cs + K_p}$$

Recommended Citation

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