Complex network analysis of forced synchronization in a hydrodynamically self-excited jet

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ARTICLE INFO

Keywords:
Flow instability
Flow control
Jets
Complex networks

ABSTRACT

Previous experiments by Li and Juniper (2013) have shown that a hydrodynamically self-excited jet can synchronize with external acoustic forcing via one of two possible routes: a saddle-node (SN) bifurcation or a torus-death (TD) bifurcation. In this study, we use complex networks to analyze and forecast these two routes to synchronization in a prototypical self-excited flow – an axisymmetric low-density jet at an operating condition close to its first Hopf point. We build the complex networks using two different methods: the visibility algorithm and the recurrence condition. We find that the networks built with the visibility algorithm are high-clustering, hierarchical, and assortative in the degree of their vertices, although only the TD networks are scale free. Nevertheless, we find that the assortativity coefficient is a sufficiently sensitive indicator by which to distinguish between the SN and TD routes to synchronization and to forecast the onset of synchronization. As for the networks built with the recurrence condition, we find that their topological features differ between the two routes to synchronization, but vary predictably along either route. We quantify these variations using statistical measures such as the mean degree, spectral radius, and transitivity dimension. This study shows that complex networks can be a useful tool for distinguishing between the SN and TD routes to synchronization, and for forecasting the proximity of a system to its synchronization boundaries. These findings could open up new opportunities for complex networks to be used in the development of open-loop control strategies for hydrodynamically self-excited flows.

1. Introduction

Open shear flows can be found in many natural and engineered processes, such as convective cooling in cross-flow heat exchangers and fuel injection in gas-turbine combustors. If an open shear flow contains a sufficiently large region of local absolute instability, it can become globally unstable, transitioning from a spatial amplifier of extrinsic perturbations to a self-excited oscillator with an intrinsic natural frequency (Chomaz et al., 1988; Huerre and Monkewitz, 1990). However, if this self-excited flow is then forced sufficiently strongly at a different frequency, it can oscillate at that frequency instead, leaving no evidence of the original natural mode (Staubli, 1987). This fully synchronous state, in which a self-excited flow oscillates only at the forcing frequency, is known as lock-in or lock-on in hydrodynamics (Williamson and Roshko, 1988; Kumar et al., 2018) or synchronization in nonlinear dynamics (Pikovsky et al., 2003; Balanov et al., 2009).

Synchronization and its industrial applications have been studied in a variety of hydrodynamically self-excited flows, particularly for the purposes of flow control. For example, detuning the natural frequency of a global hydrodynamic mode (e.g. von Kármán vortex shedding) away from that of an excitable structural or acoustic mode has been shown to be an effective means of weakening vortex-induced vibration (Blevins, 1985), aeroacoustic instabilities (Dunlap and Brown, 1981), and thermoacoustic instabilities (Poinsot et al., 1987). Furthermore, external periodic forcing has been used to control the amplitude and frequency of self-excited flow oscillations, with the aim of enhancing mixing in low-density (Hallberg and Strykowski, 2008) and cross-flow jets (Davitian et al., 2010a; 2010b; Getsinger et al., 2012), delaying transition in rotating-disk boundary layers (Pier, 2007), reducing drag from a bluff body (Menfoukh et al., 2010), and weakening vortex shedding behind a cylinder (Schumm et al., 1994). To develop improved strategies for flow control, it is thus important to be able to characterize, understand and ultimately predict the forced synchronization behavior of hydrodynamically self-excited flows.
1.1. Universal features of forced synchronization

Although the hydrodynamically self-excited flows mentioned above are physically disparate, their synchronization dynamics are remarkably similar. This is because, in all of those examples, the self-excited global mode, which arises from a sufficiently large region of local absolute instability (Chomaz et al., 1988), emerges via a Hopf bifurcation (Huerre and Monkewitz 1990). In the limit of weak nonlinearity (i.e. close to the first Hopf point where the saturated amplitude is still small), the onset, growth and forced response of a self-excited global mode are known to be well described by a generic class of low-order models based on the van der Pol (VDP) oscillator (van der Pol and van der Mark, 1927; Monkewitz, 1996; Pikovsky et al., 2003; Balanov et al., 2009). Below, we will review some of the universal features of these oscillator models when subjected to forced synchronization, drawing on examples from hydrodynamics whenever possible.

1.1.1. Synchronization boundary

A characteristic feature of forced self-excited flows is that, close to the Hopf point, the minimum forcing amplitude required for synchronization increases approximately linearly as the forcing frequency ($f_f$) shifts away from the natural frequency ($f_n$), resulting in a $\nabla$-shaped synchronization boundary centered on $f_f/f_n = 1$ (Provansal et al., 1987; Sreenivasan et al., 1989). In nonlinear dynamics, the domain within this boundary is known as the 1:1 Arnold tongue (Thompson and Stewart, 2002). Its existence has been experimentally verified in a variety of hydrodynamically self-excited flows, including low-density jets (Sreenivasan et al., 1989; Hallberg and Strykowski, 2008; Li and Juniper, 2013).

Fig. 1. The 1:1 Arnold tongue for a forced low-density jet, as adapted from Fig. 12 of Ref. (Li and Juniper, 2013a). Two different routes to synchronization are highlighted: (i) the SN route, denoted by a blue arrow and square markers, involves a saddle-node bifurcation without asynchronous quenching, whereas (ii) the TD route, denoted by a red arrow and circular markers, involves a torus-death (inverse Neimark–Sacker) bifurcation with asynchronous quenching. The vertical axis is the forcing amplitude, defined as the peak-to-peak voltage into the loudspeaker, which is directly proportional to the acoustic pressure amplitude (Li and Juniper, 2013a; 2013c). The horizontal axis is the ratio of the forcing frequency ($f_f$) to the natural frequency ($f_n$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 2. (a,c) Time traces of the normalized velocity fluctuation, and (b,d) the corresponding Poincaré sections for the two main routes to synchronization in a forced self-excited jet: (a,b) the SN route, which occurs when $f_f$ is close to $f_n$ (here $f_f/f_n = 1.04$) and involves a saddle-node bifurcation, as indicated by an abrupt collapse of the torus attractor at $A^* = 0.93 \rightarrow 1.0$; and (c,d) the TD route, which occurs when $f_f$ is far from $f_n$ (here $f_f/f_n = 1.12$) and involves a torus-death bifurcation, as indicated by a gradual shrinkage of the torus attractor at $A^* = 0.73 \rightarrow 1.0$. Along both routes, the transition from unforced periodicity to quasiperiodicity occurs via a torus-birth (TB) bifurcation. The forcing amplitude is expressed non-dimensionally as $A^* = A/A_{loc}$, where $A_{loc}$ is the minimum forcing amplitude required for complete synchronization (i.e. lock-in in Ref. Li and Juniper, 2013a). This figure is adapted from Figs. 6 and 11 of Ref. (Li and Juniper, 2013a).
Juniper, 2013a; 2013c), cross-flow jets (Davitian et al., 2010a; 2010b; Getsinger et al., 2012), capillary jets (Olinger, 1992), diffusion flames (Li and Juniper, 2013b), and cylinder wakes (Provansal et al., 1987; Olinger and Sreenivasan, 1988; Karniadakis and Triantafyllou, 1989).

1.1.2. Quasiperiodicity before synchronization

Another characteristic feature of forced self-excited flows is that, when the forcing amplitude is insufficient to cause complete synchronization and when \( f_2 \) is incommensurate with \( f_0 \) (i.e. when these two frequencies are not rational multiples of each other), the system oscillates at multiple different frequencies simultaneously (Baek and Sung, 2000). This behavior is known as quasiperiodicity (Thompson and Stewart, 2002). Quasiperiodicity appears (i) in the frequency spectrum as sharp peaks at linear combinations of \( f_2 \) and \( f_0 \), and (ii) in the phase portrait as a trajectory that spirals around the surface of a stable ergodic \( \mathbb{T}^2 \) torus attractor, never exactly repeating itself because the two incommensurate frequencies (\( f_2 \) and \( f_0 \)) produce an oscillation period of infinity (Hilborn, 2000). During quasiperiodicity, the time trace shows low-frequency modulations of its amplitude (Pikovsky et al., 2003). In forced self-excited wakes, a frequency-selection criterion has been proposed for such modulations in terms of linear combinations of \( f_2 \) and \( f_0 \) (Baek and Sung, 2000). However, surprisingly little else has been done to date to understand the detailed features of quasiperiodicity in forced self-excited flows, despite the fact that one must pass through quasiperiodicity to reach synchronization (Balunov et al., 2009).

1.1.3. Two universal routes to synchronization

Although researchers have previously investigated the universal properties of synchronization (Section 1.1.1) and, to a lesser extent, quasiperiodicity (Section 1.1.2) in forced self-excited flows, only a few of them have explored the different possible routes to synchronization. Nevertheless, identifying and understanding such routes is a key step towards being able to model and predict the synchronization dynamics of forced self-excited flows. Recent studies on low-density jets (Li and Juniper, 2013a; 2013c) and premixed combustors (Bellow et al., 2008; Balusamy et al., 2015; Kashinath et al., 2018; Guan et al., 2018a; 2018b; Guan et al., 2019) have begun to shed light on this problem through detailed analyses of the way in which a \( \mathbb{T}^2 \) torus attractor emerges from an initial unforced limit cycle, grows and then shrinks or loses stability en route to synchronization. In general, two universal routes to synchronization have been identified. To illustrate them, we show in Fig. 1 the 1:1 Arnold tongue from the forced self-excited jet studied by Li and Juniper (2013a,c). We show in Fig. 2 time traces of the normalized velocity fluctuation in the jet and the corresponding Poincaré sections. The forcing amplitude is expressed non-dimensionally as \( A^* = A/A_{loc} \) where \( A_{loc} \) is the minimum forcing amplitude required for complete synchronization. The two routes to synchronization are as follows:

(a) SN route: The SN route to synchronization occurs when \( f_2 \) is close to \( f_0 \) (Fig. 2a,b: \( f_2/f_0 = 1.04 \)) and involves a saddle-node bifurcation. When unforced (\( A^* = 0 \)), the jet oscillates periodically in a limit cycle at a natural frequency of \( f_0 \). When forced at low amplitudes (\( 0.43 \leq A^* \leq 0.57 \)), the jet becomes quasiperiodic via a torus-birth (Neimark–Sacker) bifurcation, as evidenced by the emergence of a ring in the Poincaré section. When forced at moderate amplitudes (\( 0.71 \leq A^* \leq 0.93 \)), the jet remains quasiperiodic, but its \( \mathbb{T}^2 \) torus attractor grows. When forced above a critical amplitude (\( A^* = 1 \)), the jet locks into the forcing via a saddle-node (SN) bifurcation, observed by a dramatic collapse of the phase trajectory (Fig. 2c,d: \( f_2/f_0 = 1.12 \)) and a torus-death bifurcation. When forced at low amplitudes (\( 0.33 \leq A^* \leq 0.60 \)), the jet behaves as it initially does along the SN route, transitioning from unforced periodicity to \( \mathbb{T}^2 \) quasiperiodicity via a torus-birth (Neimark–Sacker) bifurcation. As the forcing amplitude increases (\( A^* = 0.60 \rightarrow 0.93 \)), the torus attractor grows but then gradually shrinks as the synchronization boundary is approached, converging to a closed periodic orbit at \( f_2 \) when \( A^* = 1 \). This gradual shrinkage of the torus attractor is in stark contrast to the abrupt collapse along the SN route and is a key signature of a torus-death (inverse Neimark–Sacker) bifurcation to synchronization.

(b) TD route: The TD route to synchronization, by contrast, occurs when \( f_2 \) is far from \( f_0 \) (Fig. 2c,d: \( f_2/f_0 = 1.12 \)) and involves a torus-death bifurcation. When forced at low amplitudes (\( 0.33 \leq A^* \leq 0.60 \)), the jet behaves as it initially does along the SN route, transitioning from unforced periodicity to \( \mathbb{T}^2 \) quasiperiodicity via a torus-birth (Neimark–Sacker) bifurcation. As the forcing amplitude increases (\( A^* = 0.60 \rightarrow 0.93 \)), the torus attractor grows but then gradually shrinks as the synchronization boundary is approached, converging to a closed periodic orbit at \( f_2 \) when \( A^* = 1 \). This gradual shrinkage of the torus attractor is in stark contrast to the abrupt collapse seen along the SN route and is a key signature of a torus-death (inverse Neimark–Sacker) bifurcation to synchronization.

It is worth noting that only along the TD route is the oscillation amplitude significantly reduced by asynchronous quenching (Li and Juniper, 2013a). In Fig. 2(c), this can be seen by comparing the amplitude of the time trace at \( A^* = 0 \) (unforced) with that at \( A^* = 1 \) (onset of synchronization). This highlights the importance of being able to distinguish between the two routes to synchronization if one is to achieve optimal control of self-excited flows via external forcing.

1.2. Distinguishing between the two routes to synchronization

For optimal control of self-excited flows, it is important to know just how far \( f_2 \) must be tuned away from \( f_0 \) in order to cause a TD bifurcation to synchronization. This is because only along the TD route can the self-excited oscillation amplitude be significantly reduced by asynchronous quenching (Staubli, 1987; Balanov et al., 2009; Li and Juniper, 2013a; 2013c). Thus, the objective becomes one of distinguishing between the two routes to synchronization (Section 1.1.3). Conventional techniques typically rely on tracking the changes occurring in the time trace, frequency spectrum or phase space as the forcing amplitude increases at a fixed \( f_2/f_0 \) (Balunov et al., 2009). For example, the beat frequency – i.e. the frequency of the long-wavelength modulations in the oscillation amplitude – gradually approaches zero along the SN route, but abruptly snaps to zero along the TD route (Balunov et al., 2009). Furthermore, Li and Juniper (2013c) have shown that phase trapping – a partially synchronous state characterized by frequency locking without phase locking – appears only along the TD route. In phase space, the SN route can be identified by an abrupt collapse of the phase trajectory: at the onset of synchronization, the phase trajectory switches from spiraling around the surface of a stable ergodic \( \mathbb{T}^2 \) torus attractor to locking into a stable periodic orbit on that same torus surface, indicating that the underlying torus attractor has become resonant via a saddle-node bifurcation (Pikovsky et al., 2003; Balanov et al., 2009; Li and Juniper, 2013a; 2013c). By contrast, the TD route to synchronization can be identified by a gradual shrinkage of the torus attractor: in the lead up to synchronization, the phase trajectory keeps on spiraling around the surface of a stable, but shrinking, \( \mathbb{T}^2 \) torus attractor, indicating that synchronization occurs via a torus-death (inverse Neimark–Sacker) bifurcation (Pikovsky et al., 2003; Balanov et al., 2009; Li and Juniper, 2013a; 2013c).

It is recognized that distinguishing between the two routes to synchronization requires the phase trajectory or phase evolution to be delicately tracked. Although this is not difficult, there is currently no universally accepted criterion for how abruptly or gradually the phase trajectory must collapse for it to be indicative of an SN or TD route to synchronization. Therefore, this leaves the route-identification process open to misinterpretation, especially if the input data originates from a real flow system contaminated with background noise and turbulence. Consequently, there is a need for additional tools to supplement existing tools for the diagnosis and prognosis of the synchronization boundaries and the specific routes to synchronization.

1.3. Complex networks

Over the past decade, complex networks have been increasingly used to explore the multi-scale dynamics of complex systems, including those in fluid dynamics. In network-based analysis, a time series is
converted into a group of nodes connected to each other via links. Links can be created using various mathematical conditions, such as the correlation between pseudo-cycles (Zhang and Small, 2006), mutual proximity or recurrences in phase space (Nicolis et al., 2005; Donner et al., 2010), and the visibility among data points (Lacasa et al., 2008; Luque et al., 2009). Once a complex network is built, the connectivity pattern in its distribution of links can be analyzed to gain physical insight into the underlying flow behavior. For example, Gao and Jin (2009) have shown that the multiphase flow patterns appearing in gas–liquid/oil–water interfaces can produce distinct topological structures in complex networks, which can be used to quantitatively identify the transitions between different flow states. Charakopoulos et al. (2014) used time traces of the temperature in a heated turbulent jet to construct complex networks via the transformation phase-space method and the visibility graph. These researchers (Charakopoulos et al., 2014) showed that both methods produce qualitatively similar networks, and that their network topological measures can be used to distinguish between the different flow regions and dynamical states of the jet. Another useful application of complex networks is for early warning detection of undesirable events in combustion devices, such as gas turbines. For example, the groups of Sujith (Murugesan and Sujith, 2015; 2016) and Gotoda et al. (2017) have shown that complex networks can enable online forecasting of thermoacoustic instabilities and flame blowout in turbulent premixed combustors. Furthermore, Okuno et al. (2015) and Murugesan and Sujith (2015) have shown that complex networks can be used to analyze the small-world and scale-free nature of self-excited thermoacoustic oscillations, revealing high-dimensional dynamics not reported before in previous studies.

1.4. Contributions of the present study

In this study, we use complex networks to investigate the forced synchronization of a prototypical hydrodynamically self-excited flow – an axisymmetric low-density jet at an operating condition close to its first Hopf point (Li and Juniper, 2013a). To explore the two primary routes to synchronization (Section 1.1.3), we consider two different values of \( f_n \): (i) one close to \( f_n \) and (ii) the other far from \( f_n \). For different forcing amplitudes, we use the visibility algorithm and the recurrence condition to construct networks from time traces of the local streamwise velocity fluctuation in the potential core of the jet. We show that the visibility-algorithm approach can enable characterization of the high-clustering, hierarchical and assortative structure of networks arising from forced synchronization. Crucially, we find that the network topology differs depending on the specific synchronization route taken. We compute recurrence network measures – such as the mean degree, spectral radius, and transitivity dimension – and show that although these differ between the two routes to synchronization, they vary predictably along either route. In summary, this study shows that complex networks can be used as an alternative tool to distinguish between the SN and TD routes to synchronization, and to forecast the proximity of a system to its synchronization boundaries. With further development, these findings could open up new pathways for complex networks to be used in the design and testing of open-loop control strategies for hydrodynamically self-excited flows.

The rest of this paper is organized as follows. We describe the experimental setup in Section 2 and introduce quantitative measures from network analysis in Section 3. We then demonstrate the use of complex networks in distinguishing between the two routes to synchronization (SN and TD routes) in Section 4, before concluding with the key findings and implications of this work in Section 5.

2. Experimental setup

We use the forced synchronization dataset from Li and Juniper (2013a). The full details of the experimental setup and measurement procedure can be found there (Li and Juniper, 2013a), so only a brief overview is given here. The data are collected on a low-density jet discharging into quiescent ambient air at atmospheric pressure and a temperature of 293 K. The jet is created by sending helium gas into an injector fitted with a series of flow conditioning devices – i.e. a settling chamber, fine-mesh screens, and a honeycomb flow straightener – to dampen background disturbances and remove any residual non-axial velocity components from the base flow (Li, 2012). Installed at the exit of the injector is an axisymmetric convergent nozzle with a contraction area ratio of 52:1 and an exit diameter of \( D = 6 \) mm (Li and Juniper, 2013a). The convergent nozzle ensures that the initial velocity profile is top-hat in shape, with thin laminar shear layers. Additional experimental details can be found in Section 2.1 of Ref. (Li and Juniper, 2013a).

A hydrodynamically self-excited global mode is created in the jet by tuning three control parameters (Hallberg and Strykowski, 2006; Zhu et al., 2017): (i) the ratio of the density of the jet fluid to that of the surrounding air, \( S \equiv \rho_j/\rho_{amb} \approx 0.14 \); (ii) the momentum thickness of the shear layer at the nozzle exit, \( \theta_0 \), which is expressed as the transverse curvature, \( D/\theta_0 = 35.5 \); and (iii) the jet Reynolds number, \( Re = \rho_j U_j D/\mu_j = 1110 \), where \( U_j \) is the time-averaged bulk velocity and \( \mu_j \) is the dynamic viscosity of the jet fluid (helium gas at 1 atm and 293 K). This operating condition produces a self-excited global mode via a supercritical Hopf bifurcation (Li and Juniper, 2013a). The natural frequency of this mode is \( f_n = 983.0 \) Hz ± 0.15% at 95% confidence on the \( t \)-distribution. This mode appears in the jet as coherent axisymmetric oscillations of the potential core, with an azimuthal wave number of \( m = 0 \) (Monkewitz et al., 1990). These oscillations persist until \( x/D = 2.5 \) from the nozzle exit, after which secondary instabilities become dominant (Nichols et al., 2007), leading to the eruption of three-dimensional side jets, followed by a breakdown into turbulence in the far field (see Fig. 5 of Ref. Li and Juniper, 2013a).

To induce synchronization, we force the jet sinusoidally with a loudspeaker mounted at the bottom of the injector (Li and Juniper 2013a). We do this over a range of forcing frequencies \( (0.84 \leq f_n/f_m \leq 1.16) \) and forcing amplitudes \( (0 \leq A \leq 2500 \text{mV}_{pp}) \) so as to explore the full synchronization dynamics around the primary 1:1 Arnold tongue. The forcing amplitude is defined as the input voltage into the loudspeaker, which is directly proportional to the acoustic pressure amplitude (Li and Juniper, 2013a; 2013c). We measure the jet response with a hot-wire anemometer operated in constant-temperature mode, at an equivalent ratio of 1.8 (Li and Juniper, 2013a). The hot-wire probe consists of a calibrated platinum-coated tungsten wire (diameter of 5 \( \mu \)m). From the hydrodynamics literature, it is well established that the motion of a globally unstable flow is coordinated within its wave-maker region, which implies that its dynamics are predominately temporal and can be captured with measurements made at a single fixed location (Brose and Hussain, 1994). Therefore, we perform hot-wire measurements in the wave-maker region of the jet (Monkewitz et al., 1993; Chomaz, 2005; Lesshafft et al., 2006): on the jet centerline, at an axial station of \( x/D = 1.5 \) from the nozzle exit. We choose this particular sampling location because (i) it is far enough downstream for the self-excited mode to have time to grow and interact with the upstream forcing; (ii) it is upstream of the breakdown location of the self-excited mode \( (x/D \approx 2.5; \text{see above}) \); and (iii) it is still within the potential core, where the hot-wire measurements are not susceptible to contamination by fluctuations in helium concentration. Using a 16-bit data acquisition system, we digitize the hot-wire voltage at 16384 Hz for 16 s, producing a time series of the local streamwise velocity, \( u(t) \), for every combination of \( f_n \) and \( A \). This forced synchronization dataset is then analyzed using complex networks.

3. Complex network analysis

As noted in Section 1.3, various methods have been proposed for transforming time series data into complex networks. These include
methods based on mutual proximity (Nicolis et al., 2005), the visibility of nodes (Lacasa et al., 2008; Luque et al., 2009), and the recurrence of states in phase space (Donner et al., 2010). Transforming time series data into complex networks can enable information hidden in the former to be visualized as unique topological structures in the latter, providing useful physical insight into the nonlinear dynamics of the underlying flow system (Donner et al., 2010). In this study, we use two different methods – the visibility algorithm and the recurrence condition – to build complex networks, with the aim of characterizing and forecasting the forced synchronization dynamics of the jet described in Section 2.

3.1. Visibility graphs

In the visibility algorithm of Lacasa et al. (2008), observations in a time series are treated as vertices, which are linked to each other if a straight line can be drawn between them. Thus, a perfectly periodic time series appears as a regular network with ordered topological features, a stochastic time series appears as a random network, and a fractal time series appears as a scale-free network (Lacasa et al., 2008). For non-stationary signals, the power-law scaling exponents in the probability distribution of the number of links generated by the visibility algorithm are linearly related to the Hurst exponents of the associated fractal time series (Lacasa et al., 2009; Long, 2013). An extension of this method, the horizontal visibility graph, can be used to distinguish between low-dimensional and high-dimensional chaotic dynamics in the presence of noise, without the need for surrogate methods (Luque et al., 2009). In this paper, we show that networks created with the visibility algorithm using the time series of a quasi-periodic attractor can be scale free (TD route) with a high-clustering hierarchical structure. The assortativity coefficient, which is a measure of the mixing pattern in the links of a network, is shown to be capable of quantitatively distinguishing between the SN and TD routes to synchronization (Sections 3.1.3 and 3.2).

When applying the visibility condition (Lacasa et al., 2008), we treat the local maxima of our time series as vertices. Two vertices are considered to be linked if they satisfy the partial convexity constraint (Lacasa et al., 2008). Specifically, if we let $\pi(t)$ be the time series and $w(t)$ be a vector containing the local maxima of $\pi(t)$, then the individual entries in $w(t)$ are treated as vertices. Any two vertices, e.g., $w(t)$ and $w(t_l)$, are considered to be linked if all of the vertices $w(t_k)$ between $t_l < t_k < t_l$ satisfy the following condition:

$$\frac{w_j - w_k}{t_j - t_k} < \frac{w_i - w_j}{t_i - t_j}.$$  (1)

Graphically, this is shown in Fig. 3(a), where each stem (vertical blue line) represents a local maximum in $w(t)$ extracted from the time series $\pi(t)$ and is treated as a vertex (node) in the corresponding complex network.

According to the visibility condition (Lacasa et al., 2008), links are formed between any two vertices if a straight line can be drawn between the tips of their stems without intersecting any intermediate stems, as shown in Fig. 3(a). In Fig. 3(b), the vertices from Fig. 3(a) are shown as dots, and the links between them are shown as lines. Information on the links between vertices is encapsulated in the adjacency matrix, $M_{ij}$, whose elements have a value of one if two nodes ($i$, $j$) are linked together, or a value of zero otherwise (Donner et al., 2010). Networks built with the visibility algorithm are undirected and unweighted, with features that are relatively insensitive to the scales of the input time series (Donner et al., 2010).

3.2. Recurrence networks

Recurrence is a fundamental property of nonlinear dynamical systems (Marwan et al., 2007). A recurrence network is therefore a logical choice for a unified analysis of time series data based on network theory (Donner et al., 2010). The adjacency matrix of a recurrence network is defined in terms of the recurrence matrix of the input time series. For example, if $p_i$ and $p_j$ are the phase-space vectors reconstructed via time-delay embedding of the time series $x_t$, the corresponding recurrence matrix is:

$$R_{ij} = \Theta(\epsilon - ||p_i - p_j||),$$  (2)

where $\Theta(\cdot)$ is the Heaviside step function, $\epsilon$ is the recurrence threshold, and $p_i$ is the reconstructed phase-space vector at time $t_i$: $p_i = [x(t_i), x(t_i - \tau), \ldots, x(t_i - (d_i - 1)\tau)]$. Here $\tau$ is the optimal time delay, calculated as the first minimum of the average mutual information function (Fraser and Swinney, 1986). Meanwhile, $d_i$ is the minimum embedding dimension required for a one-to-one projection of the original attractor and is calculated with Cao’s method (Cao, 1997).

The adjacency matrix of a recurrence network is defined as:

$$M_{ij} = R_{ij} - \delta_{ij},$$  (3)

where the Kronecker delta function ($\delta_{ij}$) is used to remove any self-connecting vertices. In recurrence networks, each phase-space vector ($p_i$) is treated as a vertex. Vertices are linked to each other based on conditions such as the $k$-nearest phase-space vectors or a fixed phase-space distance (Donner et al., 2011b). Depending on the definition of the recurrence condition, recurrence networks can be classified as $k$-nearest neighbour networks, adaptive nearest neighbour networks, or $\epsilon$-recurrence networks (Donner et al., 2011b). In this study, we build $\epsilon$-recurrence networks using time traces of the velocity fluctuations measured in the potential core of the forced self-excited jet described in Section 2. In this framework, vertices separated by a phase-space distance of less than $\epsilon$ are considered to be linked. Similar recurrence network analyses have been used to distinguish between periodic, chaotic and stochastic dynamics in low-dimensional models (Zou et al., 2010; Subramaniyam et al., 2015), to identify dynamical transitions in a turbulent thermoacoustic system (Godavarthi et al., 2017), and to prevent lean blowout in a premixed gas-turbine combustor (Gotoda et al., 2017).

3.3. Network measures

The network topological measures quantifying the local and global structures in the distribution of links in a network can be estimated from the adjacency matrix. In this paper, we use a variety of network measures – namely the degree centrality, clustering coefficient, transitivity dimension, assortativity coefficient, characteristic path length, and spectral radius – to quantitatively analyze and forecast the dynamical changes occurring en route to synchronization. The degree centrality is a local measure used to quantify the number of links at each vertex:

$$k_n = \sum_{i=1}^{N} M_{ni}.$$  (4)

The probability that a vertex has a degree centrality of $k$ is given by the probability distribution $P(k)$. A plot of $P(k)$ versus $k$ is known as the degree distribution and can be used to characterize the topological structure of a complex network (Zhang and Small, 2006). For example, fractal scaling in a time series would produce a scale-free network with a power-law relationship of the form $P(k) \sim k^{-\gamma}$, where $2 \leq \gamma \leq 3$ (Clauset et al., 2009). The mean degree of a network, $\langle k \rangle$, is the average degree of all the nodes in that network:

$$\langle k \rangle = \frac{1}{N} \sum_{n=1}^{N} k_n.$$  (5)

The clustering coefficient of a vertex, $c_n$, is used to quantify the link density in the neighborhood of that vertex. It is defined as the ratio of (i) the number of links present in the neighborhood of a vertex $n$ to (ii) the maximum possible number of links in that same neighborhood. A
network with high clustering coefficients tends to have loop structures (triangles) near its vertices, which are different from those of a random network. Thus, a high clustering coefficient implies a unique network structure, which is linked to the cliquishness of a vertex. Watts and Strogatz (1998) defined the clustering coefficient of a vertex by counting the number of closed triangles formed by that vertex:

\[ c_v = \frac{2N_v}{k_v(k_v - 1)/2}. \]

where \( N_v \) is the number of closed triangles that vertex \( v \) participates in, and \( k_v = k_v - 1)/2 \) is the maximum possible number of triangles that the vertex can form with its neighbors. For example, \( c_v \) for a vertex with less than two links is zero because it cannot form triangles.

The global clustering coefficient of a network, \( \langle c \rangle \), is defined as the average of the clustering coefficients of all the individual vertices:

\[ \langle c \rangle = \frac{1}{N} \sum_{v=1}^{N} c_v. \]

In this definition of \( \langle c \rangle \), an equal weight is given to each vertex, even those with sparse connectivity, which contribute only a few closed triangles.

To overcome this problem, Barrat and Weigt (2000) introduced a slightly different measure, the network transitivity \( T \), which is defined as the ratio of (i) the number of triangles in a network to (ii) the number of connected triples of nodes in that network. The network transitivity and the global clustering coefficient have the same physical significance in that they both measure the relative frequency of triangles. The network transitivity can be used to define the transitivity dimension, which is a measure of the global dimensionality of an attractor (Donner et al., 2011a):

\[ D_T = \frac{\log T}{\log(3/4)}. \]

Non-integer values of \( D_T \) indicate fractal structures, which are often seen in chaotic systems. Subramaniam et al. (2015) have shown that \( D_T \), for \( \epsilon \)-recurrence networks can distinguish between low- and high-dimensional chaotic and stochastic processes, even in the presence of significant noise.

Assortative mixing in a network refers to the preference for nodes to link to others of a similar degree. This property can be quantified with the assortativity coefficient, which, for undirected networks, is defined as the Pearson correlation coefficient of the vertex degrees on both ends of a link:

\[ r = \frac{E^{-1} \sum_{i,j,k} a_{i,j,k} - [E^{-1} \sum_{i,j} (d_i + k_j)]^2}{E^{-1} \sum_{i,j} (d_i + k_j)^2 - (E^{-1} \sum_{i,j} (d_i + k_j))^2}, \]

where \( a_{i,j,k} \) are the degrees of the nodes situated at the ends of the \( k \)-th link, where with \( E \) being the total number of links in the network. Note that \( r \in [-1, 1] \), with positive values indicating assortative mixing and negative values indicating disassortative mixing. In this study (Section 4.1.3), it is found that networks created by the visibility condition are assortative, with positive values of \( r \). This occurs because of the tendency of these networks to break up into separate communities. In visibility networks, the nodes formed from periodic modulations are tightly connected to each other, resulting in higher values of \( r \). By contrast, heterogeneity in the degrees of the connected nodes leads to lower values of \( r \).

The shortest path length \( (d_{ij}) \) between any two vertices \( (i, j) \) is defined as the minimum number of links connecting those two vertices. The average value of \( d_{ij} \) over all pairs of vertices is the characteristic path length:

\[ L = \frac{1}{N(N-1)} \sum_{i,j} d_{ij}. \]

The spectral radius \( (\lambda_{\text{max}}) \) of a network is defined as the largest eigenvalue of the corresponding \( M_{ij} \):

\[ \lambda_{\text{max}} = \max(|\lambda_i|), \quad i = 1, 2, ..., N \]

where \( \lambda_i \) is the \( i \)-th eigenvalue of \( M_{ij} \).

4. Results and discussion

In this section, we perform complex network analysis of the synchronization dynamics of the forced self-excited jet studied by Li and Juniper (Li and Juniper, 2013a), with the aim of finding quantitative measures by which to distinguish and forecast the two routes to synchronization (Section 1.2).

4.1. Visibility graphs

We begin by transforming the time series data \([u'(t)/\bar{u}\), which is sampled at 16384 Hz for 16 s\] into complex networks using the visibility algorithm (Section 3.1). For both the SN and TD routes to synchronization, we show the degree distribution in Fig. 4, the clustering coefficient in Fig. 5, and the assortativity coefficient in Fig. 6.

4.1.1. Degree distribution

The degree distribution for both routes to synchronization is shown in Fig. 4. Here the power-law exponent \( (\gamma) \) is determined using the maximum likelihood method proposed by Clauset et al. (2009). We find the value of \( \gamma \) that minimizes the Kolmogorov–Smirnov goodness-of-fit statistic, as shown in Fig. 4. Along the SN route (Fig. 4a), the networks associated with quasiperiodicity follow a power-law scaling, \( P(k) \sim k^{-\gamma} \), with an exponent outside the range of \( 2 \leq \gamma \leq 3 \), indicating that these networks are not scale free and that the underlying time series do not contain fractal scaling (Clauset et al., 2009).

Along the TD route (Fig. 4b), the networks associated with quasiperiodicity also follow a power-law scaling, \( P(k) \sim k^{-\gamma} \), with an exponent inside the range of \( 2 \leq \gamma \leq 3 \), indicating that these networks are scale free and that the underlying time series contain fractal scaling (Clauset et al., 2009). To the best of our knowledge, this is the first time that scale-free behavior has been identified in a hydrodynamically self-excited flow undergoing forced synchronization. Similar scale-free behavior has been reported before in unforced turbulent flow systems (Charakopoulos et al., 2014; Murugesan and Sujith, 2015). Scale-free networks are unique in that they contain a number of highly connected nodes (known as hubs), which make up a significant fraction of the total number of links in the network. Charakopoulos et al. (2014) and...
Murugesan and Sujith (2015) have shown that these hubs are physically caused by the presence of large-scale vortices in the flow, which produce high-amplitude fluctuations in the time series.

4.1.2. Clustering coefficient

Fig. 5(a,b) shows the local clustering coefficient $c_n$ as a function of node degree $k_n$, while Fig. 5(c,d) shows the global clustering coefficient $\langle c \rangle$ as a function of node number $N$ for networks corresponding to (left) the SN route and (right) the TD route. For both routes, the local clustering coefficient follows a power-law scaling of the form $c_n(k_n) \sim k_n^{\gamma}$ (Fig. 5a,b), which suggests that the networks are hierarchical. This interpretation is consistent with Fig. 5(c,d), where $\langle c \rangle$ is invariably high.

Fig. 4. Degree distribution of complex networks constructed from the visibility algorithm using time traces of the velocity fluctuation in a forced self-excited jet during quasiperiodicity. Two different routes to synchronization are shown: (a) the SN route at the conditions of Fig. 2(a,b) and (b) the TD route at the conditions of Fig. 2(c,d). QP denotes quasiperiodicity.

Fig. 5. (a,b) Local clustering coefficient ($c_n$) as a function of node degree ($k_n$) for complex networks constructed using the visibility algorithm. Two different routes to synchronization are shown: (left) the SN route at the conditions of Fig. 2(a,b) and (right) the TD route at the conditions of Fig. 2(c,d). Also shown is (c,d) the global clustering coefficient ($\langle c \rangle$) as a function of the number of nodes ($N$). QP denotes quasiperiodicity.
and independent of the number of nodes \( N \), indicating that these networks are made up of highly connected clusters of vertices, i.e. modules. The coexistence of modular topology and scale-free behavior in the degree distribution (Section 4.1.1) can only occur if the modules are connected to each other in a hierarchical manner. In a hierarchical network, small groups of nodes are densely connected to each other to form clusters, resulting in a high clustering coefficient. The hierarchical network structure preserves the self-similar arrangement of the modules. Consequently, the fact that (i) the local clustering coefficient obeys a power-law scaling of the form \( c_L(k) \sim k^{-d} \) and (ii) the global clustering coefficient is independent of \( N \) corroborates the high-clustering hierarchical structure of these networks (Ravasz et al., 2002; Ravasz and Barabási, 2003). However, not all hierarchical networks are scale free. Along the SN route, for example, hierarchical behavior is observed without a scale-free degree distribution (Section 4.1.1). The network structure of hierarchical random graphs has been discussed in Ref. (Clauset et al., 2008). As for the physical implications of this, we note that a hierarchical network structure could imply the presence of a nested group of vortices generating similar amplitudes in the velocity fluctuations, producing a high clustering coefficient and a weaker interaction with other groups of similarly nested vortices.

4.1.3. Assortativity coefficient

As noted in Section 3.3, assortative mixing is another important measure for quantifying the pattern of connections in a complex network. For example, if vertices with high degrees tend to be connected to other vertices with high degrees, the network is said to show assortative mixing. The degree of assortative mixing can be quantified with the assortativity coefficient \( r \), which was defined in Section 3.3. In Fig. 6, we show \( r \) as a function of \( A^* \) for the two routes to synchronization. We find that \( r \) is positive for all the forcing conditions examined, indicating that the networks are assortative. This finding could be due to the presence of a community structure, i.e. groups of vertices with a high density of links within each group, but a low density of links across different groups (Newman, 2003b; 2003a). We find that the networks for quasiperiodicity along the TD route have relatively high values of \( r \), which could be related to the network nodes formed from short-wavelength modulations of the time trace. We also find that, as the forcing amplitude increases during quasiperiodicity, \( r \) increases along the TD route but decreases along the SN route. This is thought to occur because the period of the amplitude modulations increases along the SN route (Fig. 2). In visibility networks, an increase in period increases the number of network nodes participating in the formation of groups and, in turn, reduces the density of links between nodes within groups, thus reducing \( r \). At the onset of synchronization (\( A^* = 1 \)), the peaks in the time trace are no longer modulated, producing an abrupt change in the value of \( r \) along both the SN and TD routes. The marked difference in \( r \) values observed between the SN and TD routes makes this a promising metric for distinguishing between the two routes to synchronization.

4.2. Recurrence networks

We use recurrence networks to analyze the topological structures of the attractors produced by forced synchronization. In this procedure, the choice of the recurrence threshold \( \epsilon \) is crucial. If \( \epsilon \) is too small, the resulting network may break into disconnected nodes, but if \( \epsilon \) is too large, the finer details of the attractor may not be embedded in the network. In this study, we choose \( \epsilon \) using the method of Jacob et al. (2016), in which the optimal value of \( \epsilon \) is taken to be that which causes the network to be fully connected. This condition can be determined by tracking the characteristic path length \( L \) as \( \epsilon \) increases (Godavarthi et al., 2017). Fig. 7 shows \( L \) as a function of \( \epsilon \) for several quasiperiodic states along (a) the SN route and (b) the TD route. As noted in Section 3.3, \( L \) is defined as the average of the shortest path lengths between vertices. If a network contains disconnected vertices, \( L = \infty \). For both routes to synchronization (Fig. 7), when \( \epsilon \leq 0.12, L \rightarrow \infty \) for some quasiperiodic states, as indicated by the markers containing vertical arrows. In other words, when \( \epsilon > 0.12, \) the networks become fully connected, with \( L \) decreasing as \( \epsilon \) increases owing to an increase in the number of links. When \( \epsilon > 0.30, L \) becomes relatively insensitive to \( \epsilon \). According to Jacob et al. (2016), the optimal value of \( \epsilon \) should be chosen from this empirical range, 0.12 \( \leq \epsilon \leq 0.30 \).

Fig. 8 shows \( \epsilon \)-recurrence networks for the two routes to synchronization (Section 1.1.3). These networks are constructed from adjacency matrices using the open-source software package, Gephi (Bastian et al., 2009). It can be seen that the structural features of the attractors are preserved in the networks. Nodes in these networks represent phase-space vectors and are connected if the distance between

![Fig. 6. Assortativity coefficient \( r \) as a function of the normalized forcing amplitude \( A^* \) for complex networks constructed using the visibility algorithm. Two different routes to synchronization are shown: the SN route at the conditions of Fig. 2(a,b), and the TD route at the conditions of Fig. 2(c,d).](image)

![Fig. 7. Characteristic path length \( L \) as a function of the recurrence threshold \( \epsilon \) for the two main routes to synchronization in a forced self-excited jet: (a) the SN route at the conditions of Fig. 2(a,b), and (b) the TD route at the conditions of Fig. 2(c,d). QP denotes quasiperiodicity. The markers with vertical arrows are for \( L = \infty \).](image)
In Fig. 8, the nodes are colored according to their degree. The recurrence threshold is ε = 0.20. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

them is less than a critical threshold, ε = 0.20. In Fig. 8, the nodes are colored based on their degree. For both routes to synchronization, the networks for the unforced state (A* = 0) and the synchronized state (A* = 1) are seen to have cyclic attractors with high levels of recurrence. Therefore, nodes in these networks have higher degrees and are colored mostly in cooler tones (blue/purple). During quasiperiodicity, the networks along the SN route (Fig. 8–left) have similarly high levels of recurrence and are also colored in cooler tones (blue/purple). By contrast, the networks along the TD route during quasiperiodicity (Fig. 8–right) are sparse, with fewer connections, as indicated by the warmer tones (green/orange/red). These ε-recurrence networks enable us to visualize the dynamical transitions occurring during forced synchronization in terms of distinct changes in network topology. In the next subsection, we will characterize these changes using quantitative network measures.

4.3. Recurrence network measures: ⟨k⟩, λm and Dτ

In Figs. 9–11, we show the mean degree ⟨k⟩, the spectral radius λm, and the transitivity dimension Dτ as a function of A* for the two routes to synchronization. These network measures are calculated as an ensemble mean of sub-networks constructed from every 0.1831 s of the input time series, with ε = 0.15, 0.20, 0.25 and 0.30. The input time series is sampled at 16384 Hz for 16 s. The recurrence networks are constructed with a time lag of 5 and an embedding dimension of 12. We find that these ε-recurrence network measures exhibit significant changes en route to synchronization.

As Figs. 9 and 10 show, ⟨k⟩ and λm behave qualitatively similarly to each other as A* increases. In recurrence networks, ⟨k⟩ represents the global recurrence rate (i.e. the average phase-space density at a given ε), while λm is the largest eigenvalue of the adjacency matrix. When the jet is in an unforced state (A* = 0) or a synchronized state (A* ≥ 1), both ⟨k⟩ and λm exhibit relatively large values as compared with those during quasiperiodicity (0 < A* < 1). This occurs because the limit-cycle oscillations in both the unforced and synchronized states give rise to a regular cyclic network structure, with the highest number of connections between nodes. As ε increases, both ⟨k⟩ and λm increase owing to an increase in phase-space density.

Fig. 11 shows that Dτ takes on relatively low values when the jet is either unforced (A* = 0) or synchronized (A* ≥ 1), which contrasts with the trends observed in ⟨k⟩ and λm. As A* increases during quasiperiodicity (Fig. 9) and λm (Fig. 10) but higher values of Dτ (Fig. 11) relative to those of the unforced and synchronized states. Crucially, before the onset of synchronization, when 0.8 ≲ A* < 1, both ⟨k⟩ and λm along the TD route reach their minimum values and then begin to rise on the final leg to synchronization. Similarly, Dτ (only for ε = 0.30) along the TD route reaches its maximum value and then begins to fall on the final leg to synchronization. This suggests that all three network measures can serve as early warning indicators of the proximity of a system to its synchronization boundaries (A* = 1).

Finally, we note that, at the onset of synchronization, ⟨k⟩, λm and Dτ all change more abruptly along the TD route than along the SN route. This contrasts with the behavior of the torus attractor observed in the Poincaré sections (Fig. 2) but can be reconciled by considering the beat frequency (Fig. 2). As A* increases along the SN route, fτ is gradually pulled towards f0, reducing the beat frequency (f0 − fτ) smoothly (Li and Juniper, 2013a; 2013c). Such frequency pulling, however, does not occur along the TD route, where lock-in occurs via asynchronous quenching: the amplitude of the natural mode decreases until it is fully suppressed (Li and Juniper, 2013a; 2013c). Thus, the beat frequency approaches zero gradually along the SN route (Fig. 2a) but abruptly along the TD route (Fig. 2c), resulting in a similar difference in the behavior of the recurrence network measures (Figs. 9–11).

5. Conclusions

In many engineering situations, the application of external periodic forcing to a hydrodynamically self-excited flow is a common form of flow control, but there is a need for additional tools to supplement existing tools for the diagnosis and prognosis of the synchronization boundaries and the specific routes to synchronization. In this study, we use complex networks to analyze and forecast the forced synchronization dynamics of a prototypical self-excited flow – a low-density (helium) jet discharging into quiescent ambient air at an operating condition close to its first Hopf point. We focus on two classical routes to synchronization: (i) the SN route, which occurs when fτ is close to f0 and involves a saddle-node bifurcation, and (ii) the TD route, which occurs when fτ is far from f0 and involves a torus-death bifurcation. For optimal
open-loop control of self-excited flows, it is important to know how far $f_f$ must be tuned away from $f_n$ in order to induce a TD route to synchronization, because only along this specific route can the self-excited oscillation amplitude be reduced by asynchronous quenching (Li and Juniper, 2013a).

We build complex networks using two different methods: the visibility algorithm and the $\epsilon$-recurrence condition. We find that the networks built with the visibility algorithm show power-law scaling in the degree distribution, although only the TD networks are found to be scale free. On examining $c_n$, $\langle c \rangle$ and $r$, we find that both the SN and TD networks are high-clustering, hierarchical, and assortative in terms of the degree of their vertices. Crucially, we find that during

Fig. 9. Mean degree $\langle k \rangle$ of $\epsilon$-recurrence networks as a function of the normalized forcing amplitude ($A^*$) for the two main routes to synchronization in a forced self-excited jet: the SN route at the conditions of Fig. 2(a,b), and the TD route at the conditions of Fig. 2(c,d). Four different recurrence thresholds are used: (a) $\epsilon = 0.15$, (b) 0.20, (c) 0.25, and (d) 0.30. The error bars denote the standard deviation.

Fig. 10. The same as Fig. 9 but for the spectral radius ($\lambda_m$).
quasiperiodicity, as \( A^* \) increases from 0 (unforced) to 1 (onset of synchronization), \( r \) increases along the TD route but decreases along the SN route, before returning to its regular equilibrium value at the onset of synchronization (\( A^* = 1 \)). This shows that \( r \) is capable of distinguishing between the two routes to synchronization as well as forecasting the proximity to the synchronization boundaries.

We then use \( \epsilon \)-recurrence networks to represent the dynamical transitions as changes in network topology. We find that the network structure varies between the two routes to synchronization. Along the SN route, the recurrence networks are ordered cyclic attractors (unforced and synchronized states) or torus attractors (quasiperiodic state) with high levels of recurrence for all values of \( A^* \). Along the TD route, however, the recurrence networks are sparse for the quasiperiodic states, with only a few connections present. To quantify the changes occurring in these networks, we calculate three different measures: \( \langle k \rangle \), \( \lambda_\text{sp} \), and \( D_r \). We find that all three measures are capable of distinguishing between the SN and TD routes to synchronization. However, we also find that only along the TD route can these measures serve as early warning indicators, which is not a severe limitation if the goal is to reduce the self-excited amplitude by asynchronous quenching, for such a reduction can only occur via the TD route to synchronization.

In summary, this study has shown that statistical measures calculated from complex networks can be used to distinguish between the SN and TD routes to synchronization, and to forecast the proximity of a system to its synchronization boundaries. Identifying the network structure associated with forced synchronization can further aid the development of low-order models for understanding and controlling hydrodynamically self-excited flows.

Acknowledgment

We would like to thank the Research Grants Council of Hong Kong (Projects 16235716 and 26202815) for their financial support. We would also like to thank Professor Matthew Juniper for allowing us to use the forced jet data from Ref. (Li and Juniper, 2013a).

References

Dunlap, R., Brown, R.S., 1981. Exploratory experiments on acoustic oscillations driven by Social Media.