HIGH SCHOOL COURSE

ENTRANCE TEST

Course Objectives and Test Guidelines

Both high school courses are advanced courses for the high school students. By participating in this course, students will not only enhance their problem-solving skills but also gain a deeper understanding of mathematical concepts, which will be beneficial for their academic and personal growth.

This course is designed to minimize repetition and maximize exploration of a wide variety of problem-solving concepts. Students will be encouraged to spend time on each problem, understanding that the learning process is as important as the solution. The course will also focus on developing your skills in writing proofs, a crucial aspect of advanced math competitions.

The test for this course features a few complex problems. It's important to note that we're not expecting students to solve all the problems. What we're really interested in is how students approach the problems and demonstrate their problem-solving skills. Therefore, detailed solutions are expected, regardless of whether the problem was fully solved.

It's important to understand that this course will not be easy. However, by setting this expectation, we aim to prepare you to navigate through complex concepts with patience and determination. We are looking for students who are self-motivated and have strong fundamentals. While we won't be covering basic concepts in this course, such as prime factoring, we believe that with the right mindset and dedication, you can excel in this advanced course.

This course will be taught by a highly experienced and dedicated coach, who not only focuses on math but also help students in developing right mindset and habits. Her goal is to develop the best talent in Canada, and he is committed to providing the best learning experience for all students.

If you are still interested in the course then write detailed solutions for questions you can solve and **email your work as a single pdf document to info@ckstem.org before 30th July.**

1. Let a, b, c be distinct, non zero real numbers. If two fractions among

$$\frac{a^2 - bc}{a(1 - bc)}; \ \frac{b^2 - ac}{b(1 - ca)}; \ \frac{c^2 - ab}{c(1 - ab)}$$

are equal, then prove that all of these are equal, and that their common value is

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

2. Let a, b, c be distinct real numbers such that

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0.$$

Prove that

$$\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}=0.$$

3. Let a, b, c be real numbers such that $a + b + c = abc$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} = \frac{4abc}{(1-a^2)(1-b^2)(1-c^2)}.$$

4. Find all triples (x, y, z) of positive real numbers such that

$$x^2 - 3 = (y - z)^2$$
$$y^2 - 5 = (z - x)^2$$
$$z^2 - 15 = (x - y)^2$$

5. Find the prime numbers a, b, c such that

$$\frac{2b+2c+2}{30-2c-a} = \frac{2a+b}{b+c-1} = \frac{c+27}{3a+2b+4c}.$$

6. Find all real solutions to

$$x(x + 1)(x + 2)(x + 3) = 24.$$

7. Solve in real numbers

$$x - y^2 - z = \frac{1}{3} \text{ and } y - z^2 - x = \frac{1}{6}.$$

8. Find $x, y, z \in \mathbb{N}$ such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{1004}$.

9. Calculate $a = \sqrt{333...333 - 66...66}$, where there are n digits of 6 and $2n$ digits of 3, and $n \in \mathbb{N}^*$.

10. Suppose that the polynomial $P(x) = x^3 - 5x^2 + 2x + 3$ has roots a, b, c. Find the value of $(36 - a^2)(36 - b^2)(36 - c^2)$.

11. Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

12. Find the sum of all positive integers n such that when $1^3 + 2^3 + 3^3 + \ldots + n^3$ is divided by $n + 5$, the remainder is 17.

13. Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and

$Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3 and z_4 are the roots of $Q(x) = 0$ find

$P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

14. If r_1, r_2, r_3 are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+r_1}{1-r_1} + \frac{1+r_2}{1-r_2} + \frac{1+r_3}{1-r_3}.$$

15. Positive real numbers x, y, z satisfy $xyz = 10^{81}$ and

$(log_{10}x)(log_{10}(yz)) + (log_{10}y)(log_{10}(z)) = 468.$

Find $\sqrt{(log_{10}x)^2 + (log_{10}y)^2 + (log_{10}z)^2}$.

16. Let a, b be positive reals such that

$$2 + log_2 a = 3 + log_3 b = log_6(a + b).$$

Find $\dfrac{1}{a} + \dfrac{1}{b}$.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $2f(x) - 3f(-x) = x^3 + x$ for all $x \in \mathbb{R}$. Prove that f is an odd function and compute $\Pi_{k=-10}^{10} f(k)$.

16. Suppose that

$f(x+3) = 3x^2 + 7x + 4$ and $f(x) = ax^2 + bx + c$. What is $a + b + c$?

17. Find all the couples of integer numbers $(a; b)$ such that

$$8a + b \equiv 51 \pmod{100}$$
$$b \equiv 27 \pmod{100}$$
$$0 \leq a \leq 99$$
$$0 \leq b \leq 99.$$

18. Find a natural number n such that $\tau(n) = 5$ and $n - 16$ is the product of two prime numbers.

19. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49.

20. Calculate the sum

$$S = \sum_{k=1}^{n} \frac{1}{\sqrt{k + \sqrt{k^2 - 1}}}, \text{ where } n \in \mathbb{N}^*.$$

21. Compute

$$\Pi_{k=2}^{n}\left(\frac{k^3-1}{k^3+1}\right).$$

22. Let ABC be a triangle with $m(< BAC) = 90^0$. Let D, E, and F be the feet of altitude, angle bisector, and median from A to BC, respectively. If DE = 3 and EF = 5, compute the length of BC.

23. Compute the exact value of A, B, C with

a) $A = \dfrac{5\sin x - 2\cos x}{3\cos x + 4\sin x}$, if $\tan x = \dfrac{3}{4}$;

b) $B = \dfrac{3\cos x - 4\sin x}{5\sin x + 2\cos x}$, if $\cot x = \dfrac{7}{3}$.

24. Compute the exact value of each of the following expressions.

a) $\sin\left(\frac{2\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)$

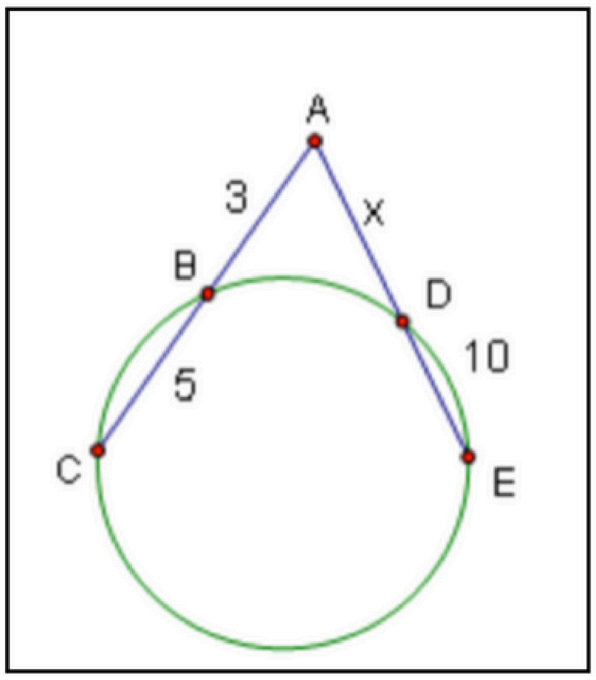
b) $\cos^2\left(\frac{2\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)$

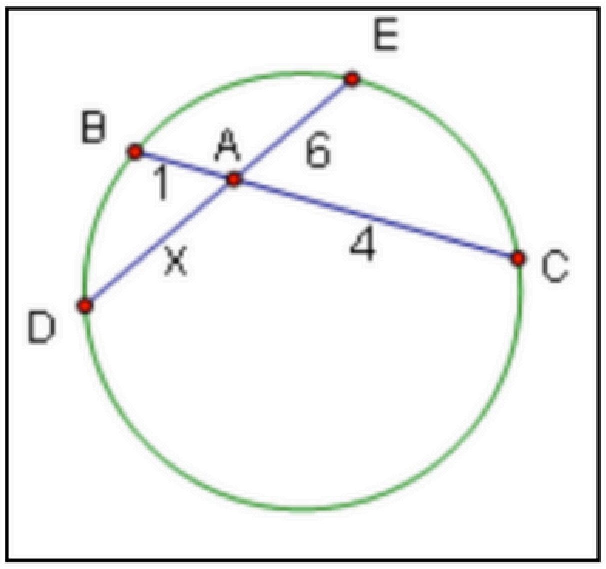
c) $\cos\left(\frac{7\pi}{6}\right)+\sin\left(\frac{5\pi}{3}\right)$

25. In triangle ΔABC, $\tan(< CAB) = \dfrac{22}{7}$, and the altitude from A divides BC into segments of length 3 and 17. What is the area of triangle ABC?

26. Prove that in any triangle ΔABC, $\dfrac{a\sin A + b\sin B + c\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \dfrac{a^2+b^2+c^2}{a+b+c}$.

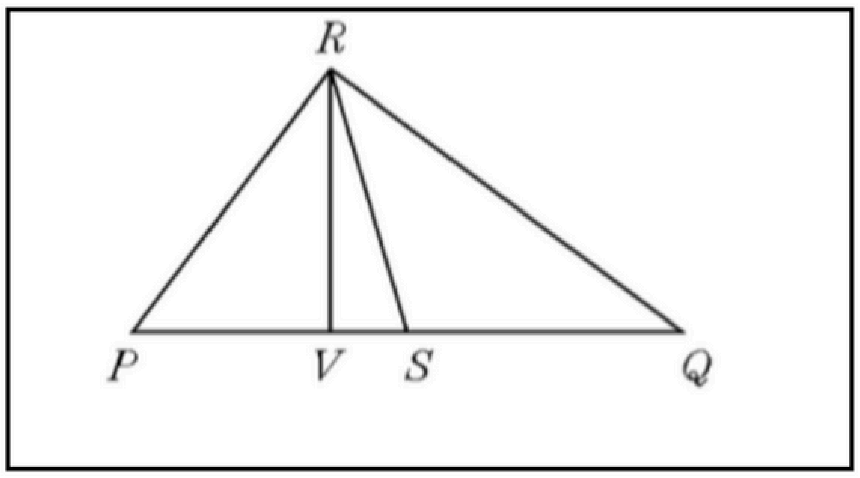
27. Find the value of x in each of the following diagrams.



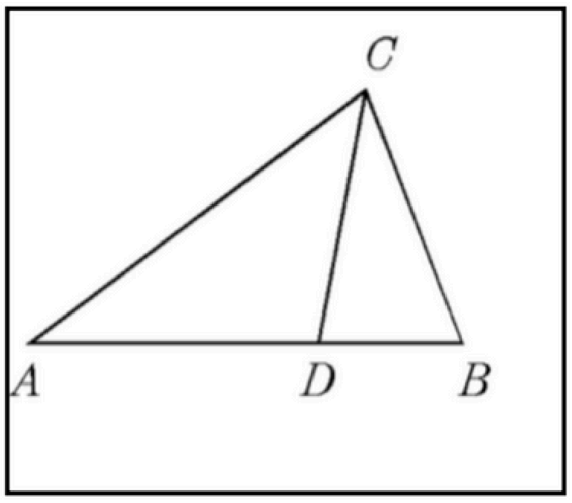


28. In unit square $ABCD$, the inscribed circle ω intersects CD at M, and AM intersects ω at a point P different from M. What is AP ?

29. In triangle ΔPQR, RV is an altitude, and RS is the angle bisector. If $PR = 15$, $RQ = 20$, and $RV = 12$, what is the area of the triangle ΔRVS in the figure ?



30. Suppose that $< ACD$ and $< DCB$ have the same measure, the length of AC is 18, the length of BC is 12, the length of AD is 9, and the length of BD is 6. What is the length of CD ?




Thank You