Competition with Complements and Substitutes*

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Abstract

We study price competition in a network setting where final products combine intermediate products supplied by (possibly) different multiproduct firms. Consumers' choices of final products follow a general demand system nesting CES and logit demands. Building on a generalized nested fixed-point argument, we develop a partially aggregative games approach to prove equilibrium existence. We then characterize equilibrium prices, profits, and consumer surplus, and provide equilibrium uniqueness conditions. Our comparative statics results illustrate two implications. Unlike in markets with pure substitutes, firm entry can raise incumbent profits and existing prices. Even under logit demand, a cost-reducing merger between suppliers of complementary products can harm consumers, but only in networks with two complementary firms, each also supplying a final product the other does not.

Keywords: Substitutes, complements, multiproduct firms, price competition, aggregative game, partially aggregative game

JEL: D43, L13, L14, L41.

1 Introduction

Multiproduct firms in many industries may produce both competing substitutes and complementary products for each other in the same market. For example, AMD pro-

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duces Central Processing Units (CPUs) that are substitutes for Intel's CPUs and Graphics Processing Units (GPUs) that are complementary to Intel's CPUs. Similarly, consider air travel between Beijing and Denver, two major cities without nonstop flight service.[1] Air China sells a codeshare itinerary bundling its own Beijing–San Francisco flight with United Airlines' San Francisco–Denver flight.[2] Meanwhile, United Airlines offers a competing itinerary flying both legs itself by bundling its own Beijing–San Francisco flight with its San Francisco–Denver flight. United Airlines is thus Air China's complementor and competitor in this city-pair market.

Beyond offering diverse product types, firms in these markets may want to charge different prices for a single product, depending on consumers' purchases of complementary products that affect their valuation of the bundle. One effective strategy is to modify a product into multiple versions, creating distinct products that can be used with different complements. Intel, for example, designed a base model but used software to disable certain features, allowing it to sell the same core hardware at different prices.[3]

When product modification is infeasible, firms may directly sell the same product at different prices, and as a result, the number of prices a firm must determine can exceed the number of physical products it produces. There are two popular arrangements. First, a firm can set prices directly based on purchases of different complementary products. Traditional examples include restaurants subsidizing parking fees or Las Vegas hotels offering room discounts contingent on active casino play. Modern technology, however, enables more complicated cross-firm arrangements. For example, Delta Air Lines (henceforth, Delta) offers flight discounts via frequent flyer miles based on the consumption of complementary products, such as taking an Uber or renting a vehicle through Hertz. These miles function as a flexible currency that can cover all or part of an airfare. By integrating reward apps across firms, Delta and Uber can easily verify consumer purchases from both firms and implement mutually contingent pricing.[4] Many other airlines and rideshare companies, such as United Airlines, Alaska Airlines, and Lyft, have also used this contingent pricing strategy, though with different terms. Second, products can be sold in bundles at different prices. For example, McDonald's often offers a price discount on its food and drinks when consumers buy both in a bundle (McCarthy (2025)). Moreover, bundling products produced by different firms is also highly prevalent in real-world markets, as illustrated by the

[1]The current nonexistence of nonstop flights between Beijing and Denver is verified on the online travel agency Expedia on August 18, 2026.

[2]A standard codeshare itinerary is an itinerary operated by more than one airline carrier but sold to customers by one of those carriers (Ito and Lee (2007), Bilotkach (2019)). Air China and United Airlines also offer similar flights with layovers in other cities (mostly Los Angeles) in this city-pair market, which are omitted for simplicity.

[3]See Maruccia (2026) for more details about Intel's software upgrade programs.

[4]Uber also provides specific discounts for Delta travelers (Delta Air Lines (2025)).

codeshare itinerary combining Air China and United Airlines flights.

One key reason firms like McDonald's offer bundle discounts is that their own products do not suffer from the double marginalization problem and may be more profitable to sell. More generally, a multiproduct firm has incentives to strategically price its products to divert consumers toward more cost-efficient and profitable product combinations (or final products). Therefore, markets with complements and substitutes raise several important questions. For example, it is unclear whether a hypothetical merger between Air China and United Airlines would necessarily reduce prices and improve consumer surplus in the Beijing–Denver market. On the one hand, the merger eliminates double marginalization, making the Air China–United Airlines combination even cheaper and more profitable to sell. On the other hand, the higher profitability may incentivize the merged firm to raise flight prices for consumers who fly both legs on United Airlines, thereby diverting them to the Air China–United Airlines combination. These opposing price effects imply that such a merger, often called a merger of complements, has an ambiguous net effect on consumer surplus.[5] Additionally, what are the effects of productivity shocks on prices and consumer surplus? How does the entry of a new competitor, which provides substitutable final products, affect the profits of incumbent firms and the prices of existing products? Due to several technical difficulties discussed below, the existing literature does not provide thorough answers to these questions. In this paper, we develop a new method called a partially aggregative games approach to solve the technical challenges and provide new answers to these questions.

We introduce a market with a network of complements and substitutes, where each final product is a combination of intermediate products supplied by (possibly) different multiproduct firms, and each final product's price is the sum of the prices of intermediate products used in this final product. Given all final products' prices, we use the demand system developed in Nocke and Schutz (2018) to model consumers' choices of the differentiated final products in a network. This demand system nests constant elasticity of substitution (CES) and multinomial logit (MNL) demands.

In a given market with complements and substitutes, we use this demand system to analyze price competition between multiproduct firms, where firms simultaneously choose prices of their intermediate products to maximize profits. We also allow firms to effectively price discriminate contingent on the purchases of different complementary products. Under MNL demand, this flexibility is generally unused in equilibrium, so our model also covers uniform pricing. In addition to the technical difficulties in analyzing price competition between multiproduct firms providing pure substitutes, such as the lack of quasi-concavity and (log-)supermodularity of the profit functions,

[5] According to the 2020 US Vertical Merger Guidelines, we define a merger of complements (or a vertical merger) as a merger between firms that provide some complementary intermediate products.

the associated pricing game is no longer a standard aggregative game.[6] Therefore, the aggregative games approach developed in Nocke and Schutz (2018), a key innovation in analyzing price competition between multiproduct firms producing pure substitutes, cannot be applied to this more general setting. Specifically, since firms may provide both complements and substitutes to each other, there does not exist a common aggregator such that a firm's profit is only affected by other firms' prices through the aggregator. However, a firm's optimal price vector still has an important property: the revised Lerner index multiplied by a measure of product elasticity is equal to a firm-level markup (we call it the η-markup). We can then treat each firm's profit maximization problem as choosing the optimal η-markup.

Based on this property, we develop a generalized nested fixed-point argument to show the existence of a pricing equilibrium. First, we use a simple method to effectively partition products in a market into components and identify a system of equations that depends on the first-order conditions of firms in the same component to determine the relationships between an aggregator and these firms' optimal η-markups. In our demand system, consumer surplus is a strictly increasing function of the aggregator. Second, the equilibrium existence problem can be further simplified to a uni-dimensional fixed-point problem, which determines the equilibrium aggregators. Finally, we recover the prices of final products and firms' profits in equilibrium from the equilibrium aggregators. Since the aggregator for each firm in our model cannot fully summarize the effects of other firms' prices, we call this new method a partially aggregative games approach. This approach also helps establish the uniqueness of a pricing equilibrium in this more general setting under MNL demand and, for a set of networks, under CES demand.

We explore the aggregator's effects on firm profits and final product prices. This analysis demonstrates how firm profits and final product prices will change as competition intensifies, particularly due to the entry of an independent competitor offering new substitutes. Our categorization of firms' complementary relationships, which can be illustrated by different networks of complements and substitutes, also plays an important role. When the network is asymmetric, where some firms with complementary products are not involved in the same final products, an increase in the aggregator may raise some firms' profits and the total prices of some final products. The reason is that as competition intensifies, some firms reduce their prices and earn lower profits, which helps other firms that provide complements to raise prices and increase profits. Since we find that the entry of a new firm, which provides substitutes for incumbent firms' products, only raises the lowest and highest equilibrium aggregators, firm en-

[6]The aggregative games are defined as the games where the payoff of each player is a function of the player's own actions and a common aggregator that summarizes all players' actions (Jensen (2018), Anderson et al. (2020)).

try can benefit incumbent firms and raise the prices of existing final products. We also find that if firms enter with sufficiently competitive products, all prices of the existing final products and all profits of the incumbent firms still go down.

Since, depending on the network type, a merger of complements may be considered a marginal-cost reduction for the merging parties' products, analyzing the impact of a reduction in marginal cost (or a quality improvement) provides direct implications regarding the effects of a merger of complements on consumer surplus. Unlike markets with pure substitutes, a cost reduction in markets with both complements and substitutes can reduce consumer surplus even with a demand like MNL. We identify the specific asymmetric networks that may generate this new result. In these networks, there exist two firms providing complementary products, each also contributing to a final product that the other does not supply. Then, a cost reduction can stimulate multiple firms to raise prices, and these price increases can reinforce each other. Overall, they can dominate the price decreases in final products with lower marginal costs and reduce total consumer surplus. When the outside option is sufficiently weak, this cost reduction may even lead to price increases in all final products, which is Edgeworth's paradox of taxation, with constant marginal costs, MNL demand, and both complements and substitutes. Hence, by the same intuition, a merger of complements can reduce consumer surplus in this type of network. In all other network types, even though a merger of complements can raise the prices of multiple final products, the merger still improves consumer surplus under both CES and MNL demands.

The rest of the paper proceeds as follows. Section 2 discusses the related literature. In Section 3, we define the network structure, the demand system, and the pricing game. In Section 4, we conduct an equilibrium analysis and show the existence of a pricing equilibrium under Marshall's second law of demand and some new conditions. We also discuss the characterization of the equilibria. In Section 5, we analyze the properties of the partially aggregative games approach. In Section 6, we perform comparative statics. In Section 7, we discuss the uniqueness of a pricing equilibrium under MNL and CES demands. In Section 8, we conclude and discuss specific applications of our model to mergers of complements and the entry of a new competitor. All proofs are relegated to the Appendix and Online Appendix.

2 Related Literature

Quint (2014a) develops a discrete choice model to analyze price competition between single-product firms in a setting with complements and substitutes. The entry of a new competitor or a merger of two firms is shown to have effects that are generally consistent with the classical models, like Bertrand or Cournot, with either complements or

substitutes. Quint (2014a) points out that the supermodular properties, which play a crucial role in the analysis, are no longer satisfied after extending the model to multiproduct firms. Several other theoretical and technical challenges, including the lack of quasi-concavity in a multiproduct firm's profit function, make the analysis of the existence and uniqueness of a pricing equilibrium with multiproduct firms even more complicated (Hanson and Martin (1996)). Not surprisingly, most research in this field has focused on single-product settings (Tan and Yuan (2003), Quint (2014b)).[7] Therefore, Quint (2014a) indicates that extending the discrete/continuous choice model to multiproduct firms in this type of market with complements and substitutes is a "significant challenge left for future work." The recent empirical studies by Luco and Marshall (2020) and Tan and Zhang (2022) provide more evidence that a merger of complements (or a vertical merger) can affect prices differently in industries with multiproduct firms and highlight the need for a comprehensive theoretical analysis in this more general setting to guide antitrust agencies. Additionally, unlike papers by Hart and Tirole (1990), Ordover et al. (1990), and Chen (2001), which analyze firms' competition and vertical mergers involving non-price responses, our analysis concentrates on firms' competition in prices, which is more consistent with the studies by Salinger (1991) and Quint (2014a).

When a single product of a firm is compatible with multiple products, a simplifying assumption that the seller cannot price discriminate based on the purchases of the complementary products is sometimes imposed (Economides and Salop (1992), Quint (2014a), Quint (2014b)). However, as Economides and Salop (1992) point out, this uniform-pricing assumption restricts the set of strategies a firm may choose and prevents a firm from offering favorable conditions to customers who purchase multiple products from this firm. Similarly, Armstrong (2013) shows that a firm has a unilateral incentive to sell a product at a different price contingent on consumers' purchase of another firm's product.[8] Allowing a single-product firm to price discriminate essentially transforms its pricing problem into a multiproduct one, which can set different prices for consumers purchasing different final products.

In recent years, a series of papers have characterized the equilibria of price compe-

[7] Two exceptions are the studies by Economides and Salop (1992) and Reisinger et al. (2021). Economides and Salop (1992) study the same pricing game under some specific networks of complements and substitutes and evaluate the impact of a merger on prices. Reisinger et al. (2021) focus on firms' incentives in building different networks of complements and substitutes. Both papers rely on the assumption of linear demand. Since we model consumers' choices of final products following Quint (2014a) and Nocke and Schutz (2018), our study does not nest linear demand.

[8] Armstrong (2013) develops a different model of commodity bundling and studies situations in which independent firms may bundle their products. In this model, consumer utility involves shocks at the component level, while our model focuses on situations where random utility shocks are only at the final-product level, consistent with consumer demands in Anderson and Leruth (1993), Quint (2014a), and Nocke and Schutz (2026). Our model may better fit situations where, for example, burger and soda are the best combination and salad and water are the second-best, rather than burger and water or salad and soda as the second-best.

tition between multiproduct firms producing pure substitutes under (nested) MNL or (nested) CES demands (Konovalov and Sándor (2010), Li and Huh (2011), Gallego and Wang (2014), Garrido (2024)). These contributions are further generalized in papers by Nocke and Schutz (2018, 2025), which build a demand system that nests both MNL and CES demands.[9] They apply the aggregative games approach, which generalizes the study by Anderson et al. (2020) with single-product firms. In more recent work, Nocke and Schutz (2026) develop a potential games approach that extends their demand system to allow products to be substitutes or complements depending on prices, including basket structures in which consumers jointly purchase products from several firms. However, in this new framework, the products of any given pair of firms are either all substitutes or all complements at any given prices. Our model instead allows two firms to produce both complementary products and competing substitutes for each other, as in the airline example with Air China and United Airlines. We extend the demand system in Nocke and Schutz (2018) to incorporate arbitrary numbers of complementary products, with no restriction on firms' ownership structures. We generalize their aggregative games approach to solve the technical challenges, show the existence of a pricing equilibrium, provide uniqueness conditions, and offer new insights on the entry of a new competitor and mergers of complements.

Our analysis also contributes to the literature on Edgeworth's paradox of taxation (Edgeworth (1925), Hotelling (1932), Salinger (1991), Armstrong and Vickers (2023)). We perform comparative statics of a reduction in one product's marginal cost, which may be caused by an exogenous tax reduction (or, similarly, a merger of complements). The existing literature studies a multiproduct monopolist and asks when a tax reduction on one product raises all of its prices (i.e., Edgeworth's paradox of taxation). With constant marginal costs, the paradox cannot arise for such a monopolist under linear or MNL demand, but it can always be generated by a suitably chosen nonlinear cost function when products are substitutes (Armstrong and Vickers (2023)). In our framework, the paradox arises with constant marginal costs and MNL demand because of the strategic interaction between firms in networks with (at least) two complementary firms, each supplying an additional final product the other does not. Moreover, our approach identifies the network structures under which such a cost reduction can improve or reduce consumer surplus, even though individual prices may move in opposite directions in both cases, as often observed in existing empirical studies (Luco and Marshall (2020), Tan and Zhang (2022)).

[9]There is a smaller literature on quantity competition between multiproduct firms (Johnson and Myatt (2006), Armstrong and Vickers (2018)). Armstrong and Vickers (2018) briefly discuss the situation where consumers may need to combine multiple complementary products. However, they do not consider cases in which multiple firms may determine the quantity or price of a combined product.

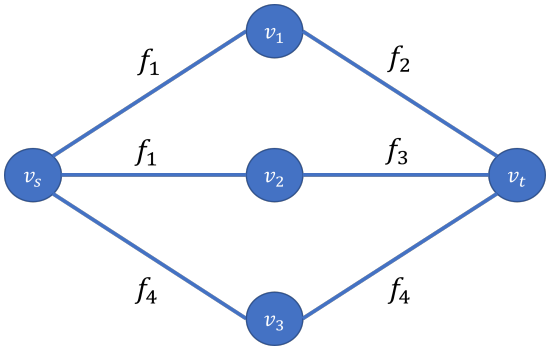


Figure 1: A network with four firms.

3 The Model

3.1 Products and Ownership Structures

We consider an economy with complementary and substitute products. There is a finite and nonempty set of firms $\mathcal{F}$, where every $f_k \in \mathcal{F}$ produces and sells intermediates that can be combined into different final products. The finite and nonempty set $\mathcal{N} = \{1, 2, ..., n\}$ denotes the final products in this economy. For each final product $i \in \mathcal{N}$, the nonempty set M_i identifies the distinct firms involved in producing intermediates for final product i. We treat M_i as part of the specification of final product i, so that two final products with the same suppliers are distinct elements of $\mathcal{N}$. Additionally, the set $G^k \equiv \{i \in \mathcal{N} : f_k \in M_i\}$ for firm $f_k \in \mathcal{F}$ identifies the final products that include f_k's intermediate products. The pair $(\mathcal{F}, \mathcal{N})$ thus determines $(M_i)_{i\in\mathcal{N}}$ and $(G^k)_{f_k\in\mathcal{F}}$, and we call it a network of complements and substitutes.

The ownership structure and the relationships among intermediate products can be visualized as an airline network between an origin (v_s) and a destination (v_t). As depicted in Figure 1, each directed path from v_s to v_t constitutes a final product, formed by a combination of intermediate products supplied by (possibly) distinct firms.[10] These paths represent competing final products within the economy. Other network features independent of $(\mathcal{F}, \mathcal{N})$ (e.g., the sequence of firms along a path) will not affect our equilibrium analysis.

Finally, we introduce the key concept of a separable component, which will be useful for our partially aggregative games approach discussed in Section 4.

Definition 1. *A separable component of the economy, $S_\alpha \subseteq \mathcal{N}$, is a nonempty minimal set of final products such that for any final product $i \notin S_\alpha$, there does not exist $f_k \in \mathcal{F}$, where $i \in G^k$ and $G^k \cap S_\alpha \neq \varnothing$.*[11]

[10] To simplify the presentation, we do not formalize this path representation involving vertices, links, and firms' ownership of the links. This can be developed in standard graph-theory language, and only the sets $(M_i)_{i\in\mathcal{N}}$ it induces matter for our analysis.

[11] Given a collection of sets, a minimal set is defined as a set that does not contain any other set from

Figure 1 illustrates an economy with two separable components, where we order the final products $\mathcal{N} = \{1, 2, 3\}$ from top to bottom.[12] These separable components are $S_1 = \{1, 2\}$ and $S_2 = \{3\}$. The paths in any economy $(\mathcal{F}, \mathcal{N})$ can be uniquely partitioned into a finite number of separable components, and we denote the set of separable components by S, where $|S| \geq 1$.[13] Then, for any $\alpha \in \{1, 2, 3, ..., |S|\}$, $S_\alpha \in S$ is a separable component of the economy. Let $L_\alpha = \bigcup_{i \in S_\alpha} M_i \subseteq \mathcal{F}$ denote the firms that provide intermediates for some or all final products in separable component S_α. Then, (L_α, S_α) gives a more complete description of the separable component S_α.

An interpretation of the set of separable components is that firms from different separable components are producing pure substitutes, and firms inside a separable component may produce complements for some other firms in this separable component. This concept of separable components is also connected to the international airline industry. We can interpret each separable component as an airline alliance, where carriers within the same alliance may provide complements to each other, and flights across alliances are often substitutable.

Our network structure is a generalization of products in Quint (2014a) and Nocke and Schutz (2018). When $|G^k| = 1$ for any $f_k \in \mathcal{F}$ and $|M_i| \geq 1$ for any $i \in \mathcal{N}$, the network corresponds to the baseline model of Quint (2014a), where single-product firms may produce complements or substitutes for other firms. In this case, each final product is a separable component. When $|G^k| \geq 1$ for any $f_k \in \mathcal{F}$ and $|M_i| = 1$ for any $i \in \mathcal{N}$, the network illustrates the market structure in Nocke and Schutz (2018), where multiproduct firms can only produce substitutes for each other. In this case, each $G^k \subseteq \mathcal{N}$ for any $f_k \in \mathcal{F}$ is a separable component. We generalize the settings in these two papers by allowing each firm to produce intermediate products for multiple final products and every final product to be composed of multiple intermediate products.

3.2 Consumer Demand

Consider an economy with complements and substitutes given by a network $(\mathcal{F}, \mathcal{N})$. Each consumer in this economy first decides which final product to buy and then the quantity to purchase. It can be interpreted as choosing the optimal path and the frequency of trips for consumers to go from the source node v_s to the destination node

the collection as a strict subset.

[12] Unless otherwise noted, paths in $\mathcal{N}$ in the remaining figures are also ordered from top to bottom.

[13] We can partition final products of an economy into separable components by a simple algorithm: Pick an arbitrary final product i to form a set S_1; add every final product $j \in \mathcal{N} \setminus \{i\}$ which shares at least one firm with final product i (i.e., $M_i \cap M_j \neq \varnothing$); expand S_1 by adding more final products that share at least one firm with a final product in S_1; repeat until no new final product can be added; pick a final product outside S_1 and repeat previous steps to get S_2; finally, stop if the set of separable components $S = \{S_1, S_2, ..., S_q\}$ has the property that $\bigcup_{\alpha=1}^{q} S_\alpha = \mathcal{N}$.

v_t. We also assume that consumers have an outside option, denoted 0.[14] Let P_i denote the price of final product $i \in \mathcal{N}$ and suppose that $P = (P_i)_{i\in\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}_{++}$ is a given price vector of final products in $\mathcal{N}$. Each final product $i \in \mathcal{N}$ delivers an indirect utility of $\log h_i(P_i) + \epsilon_i$, where $h_i(P_i)$ is a $\mathcal{C}^3$ and positive-valued function and ϵ_i represents the taste shock of final product i for each consumer. The outside option 0 offers an indirect utility of $\log H^0 + \epsilon_0$, where H^0 is a constant greater than zero and ϵ_0 is each consumer's taste shock of the outside option.[15] Conditional on having chosen final product $i \in \mathcal{N}$, each consumer consumes $-(\log h_i(P_i))'$ units of final product i, as given by Roy's identity. We assume that each consumer's taste shocks $(\epsilon_i)_{i\in\mathcal{N}\cup\{0\}}$ are drawn i.i.d. from a type-I extreme value distribution, and we can determine the probability of choosing final product i to be

$$\text{Prob}_i(P) = \frac{h_i(P_i)}{H(P)},$$

where $H(P) \equiv \sum_{j\in\mathcal{N}} h_j(P_j) + H^0$.[16] Then, the average demand per consumer for final product $i \in \mathcal{N}$ is given by

$$D_i(P) = \frac{h_i(P_i)}{H(P)} \frac{-h_i'(P_i)}{h_i(P_i)} = \frac{-h_i'(P_i)}{H(P)} = \hat{D}_i(P_i, H(P)). \tag{1}$$

We assume that for any $i \in \mathcal{N}$, $h_i(P_i)$ has the properties that $h_i' < 0$ and h_i is log-convex.[17] Note also that consumer surplus is given by $\log H(P)$. This demand system, which builds on early research on discrete/continuous choice models (e.g., Hanemann (1984), Anderson et al. (1992)), is formally introduced in the paper by Nocke and Schutz (2018), where they also show that this demand system can be derived from quasi-linear utility maximization. CES demand (with $h_i(P_i) = a_i P_i^{1-\sigma}$, where $a_i > 0$ and $\sigma > 1$) and MNL demand (with $h_i(P_i) = \exp\left(\frac{a_i - P_i}{\lambda}\right)$, where $a_i \in \mathbb{R}$ and $\lambda > 0$) are special cases of the demand system.

3.3 Pricing Game

Firms determine the prices of their intermediate products in this economy $(\mathcal{F}, \mathcal{N})$. Let s_i^k denote the total price of the intermediate products offered by firm f_k for final

[14] To simplify the presentation, the outside option has been omitted from all figures in this paper.

[15] Unlike Nocke and Schutz (2018), we do not consider the scenario where $H^0 = 0$. We need to study networks that allow a single firm to supply intermediates for all final products in $\mathcal{N}$. In this scenario, a finite pricing equilibrium does not exist with $H^0 = 0$. Allowing for infinite prices will also lead to an inconsistent interpretation of infinite prices.

[16] The probability of choosing the outside option is given by $\text{Prob}_0(P) = \frac{H^0}{H(P)}$. However, the demand for the outside option is undefined.

[17] As shown in Nocke and Schutz (2018), these assumptions can ensure that the demand system and the indirect subutility functions are valid.

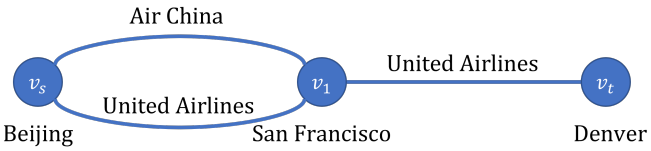


Figure 2: A network with a common intermediate product.

product i.[18] The profile of prices $(s_i^k)_{f_k \in M_i} \in \mathbb{R}_{++}^{|M_i|}$ contains the intermediate product prices charged by firms $M_i \subseteq \mathcal{F}$ for final product $i \in \mathcal{N}$. We assume that the price for any final product $i \in \mathcal{N}$ is determined by the following equation:

$$P_i \equiv \sum_{f_k \in M_i} s_i^k. \tag{2}$$

This means that the price of any final product in this network is the sum of the prices of its intermediate products.

We now define the profit of each firm in the economy $(\mathcal{F}, \mathcal{N})$ under the demand system, $((h_i)_{i\in\mathcal{N}}, H^0)$. The marginal cost of firm $f_k \in \mathcal{F}$ in producing intermediate products for final product $i \in G^k$ is given by $c_i^k \in \mathbb{R}_{++}$, where, as defined before, $G^k \subseteq \mathcal{N}$ is the set of final products that use firm f_k's intermediate products. We define $s^k = (s_i^k)_{i\in G^k}$ as a profile of intermediate product prices charged by firm f_k for final products G^k. As discussed before, we can partition any economy into a finite set of separable components S. Suppose that firm f_k provides intermediates for some final products in $S_\alpha \in S$. We use $f_{k'}$ to denote other firms providing complements for firm f_k in S_α (i.e., $G^l \cap G^k \neq \varnothing$ for every $f_l \in f_{k'}$ and $f_k \notin f_{k'}$), and $s^{k'} = (s_i^l)_{f_l \in f_{k'}, i\in(G^l \cap G^k)}$ is a vector of prices of intermediates produced by $f_{k'}$ that are complementary to intermediates provided by f_k. Moreover, $s = (s_i^k)_{f_k\in\mathcal{F}, i\in G^k}$ is a vector of intermediate product prices charged by all firms in $\mathcal{F}$, and $Q = \sum_{i\in\mathcal{N}} |M_i|$ is the total number of intermediate product prices. Then, we can use equation (2) to replace final product prices and define the profit of firm $f_k \in \mathcal{F}$ as follows:

$$\Pi^k(s^k, s^{k'}, H(s)) = \sum_{j\in G^k} (s_j^k - c_j^k)\hat{D}_j(s_j^k + \sum_{f_l \in (M_j \setminus \{f_k\})} s_j^l, H(s)), \quad s \in (0,\infty)^Q. \tag{3}$$

We assume that each firm can sell a common intermediate product (i.e., the same intermediate used in more than one final product) at different prices. In the context of Figure 2, this implies that the price of an air ticket between San Francisco and Denver is contingent upon the consumer's selection of an airline carrier for traveling between Beijing and San Francisco, as in the contingent-pricing assumption of Hendricks et al. (1997). When the two itineraries have different customer valuations or costs, this as-

[18] Because we consider a simultaneous pricing game, as specified in this section, we can assume, without loss of generality, that each firm chooses only one price for its intermediates in a final product.

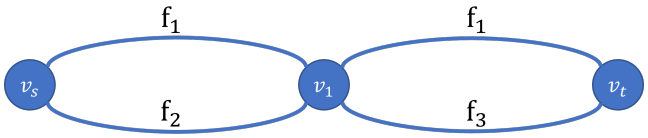


Figure 3: A network with complementary common intermediate products.

sumption allows United Airlines to easily adjust prices to maximize profit.[19] This practice also appears in the joint reward programs offered by Delta and Uber, as we discussed earlier, and reflects broader patterns in other industries. We also note that the potential feasibility issue with price discrimination for a common intermediate product, illustrated in Armstrong (2013), is much weaker in our framework. In the mixed bundling literature, if a firm charges a premium for the bundle, consumers could buy the standalone products separately and assemble them, raising the question of whether a firm can charge a premium for a bundle. However, as shown in the network illustrated in Figure 3, any individual product in the (f_1, f_1) bundle must be combined with some other intermediate product, and in real-world markets, consumers normally cannot buy bundles (f_1, f_3) and (f_2, f_1), throw away the unneeded components, and assemble the (f_1, f_1) bundle. For example, a consumer buying a codeshare itinerary with multiple flights cannot just choose one flight, because airlines cancel the remaining flights if an earlier portion of the itinerary is missed. Similarly, Delta's discount on Uber requires consumers to complete the Uber ride, preventing them from using this discount with some other ground transportation. Wireless carriers like AT&T often bundle their mobile network services with new phones, and consumers who receive the bundle discount cannot use those phones on other carriers' networks. Therefore, contingent pricing for a common intermediate product can be implemented much more easily in the markets we study. In Section 4, we highlight a feature of our model with MNL demand that extends our analysis to markets subject to the uniform-pricing restriction on a common intermediate product.

Another key assumption we make is that firms cannot set infinite prices. Allowing infinite prices means that a firm can withdraw its intermediate products to eliminate the corresponding final products. However, we want to focus on firms' pricing strategies and treat the products as exogenous. A more complete analysis could be done in a two-stage game in which firms first decide on which products to supply and then compete in prices. This paper's equilibrium analysis corresponds to the second stage of such a game. Moreover, allowing infinite prices may lead to the trivial "no-trade"-type equilibria, where one final product may have multiple infinite intermediate product

[19] We assume United Airlines charges one price to consumers who choose United Airlines for both legs. Potentially, the price of the San Francisco–Denver ticket in the codeshare itinerary can be higher than the price United Airlines effectively charges for that segment when a consumer buys both legs from United Airlines. Then, the uniform pricing of a common intermediate product may unreasonably restrict firms' pricing decisions, as discussed in Economides and Salop (1992).

prices set by several firms (Quint (2014a)). Therefore, we focus on finite prices, which will also help us generalize the standard aggregative games approach in Section 4.

We study a pricing game in which firms simultaneously set prices for their intermediate products in a given economy with a network $(\mathcal{F}, \mathcal{N})$ to maximize their respective profits. Firm profits are given by equation (3). A pure-strategy Nash equilibrium of this pricing game is defined as a pricing equilibrium.

Note that the simultaneous-pricing assumption in a market with complements and substitutes is widely used in modeling consumer purchases of complementary products (see Economides and Salop (1992) and Linnemer (2022) for analyses based on Cournot's classical model), mixed bundling (see Armstrong (2013) and Zhou (2021)), and airline pricing without antitrust immunity (see Hendricks et al. (1997), Brueckner (2001), Anderson et al. (2010)). Moreover, our framework can be viewed as a reasonable model of general vertical chains or retail competition. Following the change of variables illustrated by Tirole (1988), Quint (2014a), and Linnemer (2022), a retailer's decision regarding a final product's price can be equivalently reinterpreted as choosing the price of the retailer's inputs comprising that final product, where the final product price remains the sum of all input prices. Then, the primary difference between our model and retail competition lies in the assumption of simultaneous versus sequential pricing. In many real-world situations, simultaneous pricing is sufficient to capture firms' main strategic choices, and our simultaneous pricing game is used in many standard vertical contexts (e.g., Quint (2014a), Choi et al. (2017), and Reisinger et al. (2021)). When the downstream sector that assembles and sells the final products is perfectly competitive, as for the product aggregators in Armstrong (2013) and the manufacturers in Shapiro (2000) and Quint (2014b), the price of each final product is the sum of its intermediate product prices, which exactly corresponds to our model.

4 Equilibrium Analysis: A Partially Aggregative Games Approach

As discussed in Quint (2014a) and Nocke and Schutz (2018), the pricing game here with multiproduct firms presents similar difficulties in proving the existence of a pricing equilibrium.[20] More importantly, since multiproduct firms may produce both complements and substitutes in this more general setting, the standard aggregative games approach, reviewed in Jensen (2018), cannot be directly applied.

[20] First, Π^k in equation (3) is not necessarily quasiconcave in $(s_j^k)_{j\in G^k}$ (Hanson and Martin (1996)). Second, if f_k is a multiproduct firm (i.e., $|G^k| > 1$), then Π^k is neither supermodular nor log-supermodular in $(s_j^k)_{j\in G^k}$. In the case without complementary products, Gallego and Wang (2014) show that the pricing game is supermodular in markups. However, such a property no longer holds since the markups for complementary products may be strategic substitutes.

To solve these problems, we first define a new markup closely related to a firm's optimal price vector. Based on firms' markups, we can easily define the aggregate markup for each final product in the network. Second, we show that using (possibly) multiple equations for firms within the same separable component, each firm's markup and each final product's aggregate markup are functions of the aggregator, $H(P)$. Third, even though the aggregator for each firm summarizes only part of the effects of other firms' prices, we can still develop a similar nested fixed-point argument, which is the foundation of the standard aggregative games approach, to show the existence of a pricing equilibrium. We call this new method a partially aggregative games approach.[21] In the following, we provide a brief and informal discussion of our new approach, and interested readers can refer to the Appendix for details.

For firm $f_k \in \mathcal{F}$, it chooses s^k to maximize its profit, which is given by equation (3). To simplify the exposition, we suppose for the moment that the first-order conditions are both necessary and sufficient for the optimality of each firm's profit maximization problem. The first-order conditions hold at price vector s if and only if, for every $f_k \in \mathcal{F}$ and $i \in G^k$,

$$\begin{aligned}\frac{\partial \Pi^k}{\partial s_i^k} &= \hat{D}_i + (s_i^k - c_i^k)\frac{\partial \hat{D}_i}{\partial P_i} + \frac{\partial H}{\partial s_i^k}\Big(\sum_{j \in G^k}(s_j^k - c_j^k)\frac{\partial \hat{D}_j}{\partial H}\Big), \\ &= \hat{D}_i\Big(1 - \frac{s_i^k - c_i^k}{P_i}\Big|\frac{\partial \log \hat{D}_i}{\partial \log P_i}\Big| + \frac{\frac{\partial H}{\partial s_i^k}}{\hat{D}_i}\Big(\sum_{j \in G^k}(s_j^k - c_j^k)\frac{\partial \hat{D}_j}{\partial H}\Big)\Big) = 0.\end{aligned}$$

The first-order condition can be rewritten as

$$\frac{s_i^k - c_i^k}{P_i}\Big|\frac{\partial \log \hat{D}_i}{\partial \log P_i}\Big| = 1 + \sum_{j \in G^k}(s_j^k - c_j^k)\frac{\partial \hat{D}_j}{\partial H}\frac{\frac{\partial H}{\partial s_i^k}}{\hat{D}_i}.$$

Let $\eta_i(P_i) = P_i\frac{h_i''(P_i)}{-h_i'(P_i)} = \big|\frac{\partial \log \hat{D}_i}{\partial \log P_i}\big|$. The equation above is equivalent to

$$\frac{s_i^k - c_i^k}{P_i}\eta_i(P_i) = 1 + \Pi^k(s^k, s^{k'}, H(s)). \tag{4}$$

The price vector s is a pricing equilibrium if and only if s satisfies equation (4) for every $f_k \in \mathcal{F}$ and $i \in G^k$. Equation (4) has the property that the right-hand side is the same for every $i \in G^k$. Unlike in Nocke and Schutz (2018), the total price P_i on the left-hand side of equation (4) depends on prices charged by other firms producing complements for f_k in final product i. In any pricing equilibrium, for any $f_k \in \mathcal{F}$ and

[21] The fully aggregative games are defined as the games where the payoff and the marginal payoff of a player are functions of the player's own actions and a common aggregator that summarizes all players' actions (Cornes and Hartley (2012)).

for any $i, j \in G^k$, we must have

$$\frac{s_i^k - c_i^k}{P_i}\eta_i(P_i) = \frac{s_j^k - c_j^k}{P_j}\eta_j(P_j).$$

We define firm f_k's η-markup as μ^k, where $\mu^k = \frac{s_i^k - c_i^k}{P_i}\eta_i(P_i) > 1$ for every $i \in G^k$, since equilibrium profits are strictly positive, and we say that firm f_k's optimal profile of prices, s^k, satisfies the common η-markup property. Equation (4) also implies that firm f_k's equilibrium profit equals its equilibrium η-markup minus one.

The appearance of other firms' prices in a firm's η-markup resembles the adjustment of the inverse elasticity rule for markets with complementary products, as shown in Hendricks et al. (1997). Therefore, we can interpret the common η-markup property as in models without complementary products (Nocke and Schutz (2018)). Under monopolistic competition, where firm $f_k \in \mathcal{F}$ does not internalize the impact of its prices on the aggregator H, the demand for final product $i \in G^k$ is $\hat{D}_i(P_i, H)$ for an $H > 0$, and the price elasticity of demand for i is $|\frac{\partial \log \hat{D}_i}{\partial \log P_i}| = \eta_i(P_i)$. Then, firm f_k's optimal price for its intermediates in final product i satisfies the inverse elasticity rule with complements: $\frac{s_i^k - c_i^k}{P_i} = \frac{1}{\eta_i(P_i)}$. In our model, firm f_k internalizes its impact on the aggregator H and chooses a higher s_i^k when prices $(s_i^l)_{f_l \in M_i \setminus \{f_k\}}$ are the same, according to the equation $\frac{s_i^k - c_i^k}{P_i} = \frac{\mu^k}{\eta_i(P_i)}$ with $\mu^k > 1$.

Another implication of the common η-markup property is that under MNL demand, $\mu^k = \frac{s_i^k - c_i^k}{P_i}\eta_i(P_i) = \frac{s_i^k - c_i^k}{\lambda}$ for every $i \in G^k$ and $f_k \in \mathcal{F}$. If firm f_k only provides a common intermediate product for final products i and j, then $c_i^k = c_j^k$, which implies that $s_i^k = s_j^k$ by the common η-markup property. Therefore, under MNL demand, the producer of a common intermediate product does not have an incentive to price discriminate based on complementary products. This explains why the uniform-pricing assumption for a common intermediate product in Quint (2014a) and Quint (2014b) with single-product firms and MNL demand is actually nonbinding. The only subtle issue in our framework lies in the assumption that a firm charges only one total price for its intermediate products in a final product. Consider a specific case where a firm produces two complementary common intermediate products, as illustrated in Figure 3. Firm f_1's products can all be used in more than one final product but not in all of them, and f_1 chooses the same absolute markup whether it sells both products or only one under MNL demand. As a result, firm f_1 must offer a price discount when a consumer buys both of its products. This popular and easy-to-implement pricing strategy is the minimal departure from strict uniform pricing of a common intermediate product in our model. In other words, by allowing a firm to set one unconstrained total price for its intermediate products in a final product, our model with MNL demand extends the analysis of Quint (2014b) to multiproduct firms.

Next, we provide more details about why the standard aggregative games approach cannot be applied and how we solve the technical challenges. For any final product $i \in \mathcal{N}$,

$$\sum_{f_l \in M_i} \mu^l = \sum_{f_l \in M_i} \frac{s_i^l - c_i^l}{P_i} \eta_i(P_i) = \frac{P_i - C_i}{P_i} \eta_i(P_i) \equiv \Psi_i, \tag{5}$$

where $C_i \equiv \sum_{f_l \in M_i} c_i^l$ is the sum of all firms' marginal costs in providing final product $i \in \mathcal{N}$. We call Ψ_i the aggregate markup for final product i. Suppose for the moment that the function $P_i \mapsto \frac{P_i - C_i}{P_i} \eta_i(P_i)$ is one-to-one for any $i \in \mathcal{N}$. We denote its inverse function by $t_i(\cdot)$, which is called the pricing function for final product i. Let $\gamma_i(P_i) = \frac{h_i'^2(P_i)}{h_i''(P_i)}$ for any final product $i \in \mathcal{N}$. We can use η-markups and aggregate markups to simplify firms' profit maximization problems by rewriting equation (4) for any $f_k \in \mathcal{F}$ to be

$$\begin{aligned} \mu^k =& 1 + \mu^k \frac{\sum_{j \in G^k} \gamma_j(t_j(\Psi_j))}{H}, \quad \text{where} \\ H =& \sum_{j \in \mathcal{N}} h_j(t_j(\Psi_j)) + H^0. \end{aligned} \tag{6}$$

Equation (6) shows that the standard aggregative games approach may not be directly applied to this more general setting. The right-hand side of equation (6), which is firm f_k's profit plus one, depends on both the aggregator H and the η-markups of firms providing complements for firm f_k in its separable component S_α (i.e., $G^k \subseteq S_\alpha$). As shown in Figure 1, two firms providing complementary intermediate products may also be involved in different final products. Then, if $G^k \cap G^l \neq \varnothing$ and $G^k \neq G^l$ for $f_k, f_l \in \mathcal{F}$, we may have $\sum_{j \in G^k} \gamma_j(t_j(\Psi_j)) \neq \sum_{j \in G^l} \gamma_j(t_j(\Psi_j))$. This implies that for some firms' profits, there does not exist a common term that summarizes the impact of all firms' prices or η-markups. More importantly, unlike Nocke and Schutz (2018) or Anderson et al. (2020), the standard fitting-in function for a firm (i.e., a function from a common term to this firm's best response), pinned down by this firm's profit maximization problem, cannot be identified.[22]

Instead, we pick a system of equations (6) for firms from the same separable component S_α, where the corresponding set of firms is L_α. With a given aggregator $H \geq H^0$, this system of equations has a unique solution in $(\mu^k)_{f_k \in L_\alpha}$ according to the generalized Poincaré-Hopf index theorem in Simsek et al. (2007). We denote this solution by $(g^k(H))_{f_k \in L_\alpha}$, and we call $g^k(H)$ firm f_k's fitting-in function.

By equation (5), we now know that the aggregate markup for final product $i \in S_\alpha$ can also be considered a function of the aggregator H, which we denote $m_i(H)$. This

[22] The pricing game is also not quasi-aggregative in general (Jensen (2010), Jensen (2018)). Quasi-aggregation requires each firm's profit to depend on rivals' prices through a single real-valued function, and no such function generally exists when multiproduct firms produce complements and substitutes.

function is called final product i's fitting-in function. Similarly, $g^l(H)$ for any firm $f_l \notin L_\alpha$ or $m_j(H)$ for any final product $j \notin S_\alpha$ can be determined by equations (6) for the corresponding separable component. Therefore, the equilibrium existence problem becomes finding an aggregator H such that

$$H = H^0 + \sum_{j \in \mathcal{N}} h_j(t_j(m_j(H))) \equiv \Gamma(H), \tag{7}$$

where $\Gamma(H)$ is the aggregate fitting-in function. Since the common aggregator H for each firm only summarizes the impact of part of the other firms' prices or η-markups, as shown in equation (6), our new method is only partially aggregative.

4.1 An Illustrative Example with MNL Demand

To better understand our new method, we study the pricing game in the network with two separable components shown in Figure 1. We also focus on an MNL demand specification with $h_i(P_i) = \exp\left(\frac{a_i - P_i}{\lambda}\right)$ for every $i \in \mathcal{N}$. As firm f_1 offers intermediate products for final products 1 and 2, f_1's profit is given by

$$\Pi^1(s^1, s^{1'}, H(s)) = (s_1^1 - c_1^1)\hat{D}_1(s_1^1 + s_1^2, H(s)) + (s_2^1 - c_2^1)\hat{D}_2(s_2^1 + s_2^3, H(s)).$$

Then, the first-order conditions for firm f_1's profit maximization problem show that

$$0 = \frac{\partial \Pi^1}{\partial s_i^1} = \hat{D}_i(1 - \frac{s_i^1 - c_i^1}{\lambda} + \Pi^1), \quad \forall i \in \{1, 2\}.$$

The equation above allows us to reproduce equation (4), and we can see that firm f_1's η-markup satisfies the common η-markup property: $\frac{s_1^1 - c_1^1}{\lambda} = \frac{s_2^1 - c_2^1}{\lambda} = \mu^1 = 1 + \Pi^1$.

We repeat the analysis above and rewrite equation (6) for all firms. The equilibrium η-markups are determined by the following equations:

$$\tau^1(\mu^1, \mu^2, \mu^3, H) = 1 - \frac{1}{\mu^1} - \frac{h_1(t_1(\mu^1 + \mu^2)) + h_2(t_2(\mu^1 + \mu^3))}{H} = 0, \tag{8a}$$

$$\tau^2(\mu^1, \mu^2, H) = 1 - \frac{1}{\mu^2} - \frac{h_1(t_1(\mu^1 + \mu^2))}{H} = 0, \tag{8b}$$

$$\tau^3(\mu^1, \mu^3, H) = 1 - \frac{1}{\mu^3} - \frac{h_2(t_2(\mu^1 + \mu^3))}{H} = 0, \tag{8c}$$

$$\tau^4(\mu^4, H) = 1 - \frac{1}{\mu^4} - \frac{h_3(t_3(\mu^4))}{H} = 0, \tag{8d}$$

$$H = h_1(t_1(\mu^1 + \mu^2)) + h_2(t_2(\mu^1 + \mu^3)) + h_3(t_3(\mu^4)) + H^0. \tag{8e}$$

In this case with MNL demand, $t_i(\Psi_i) = \lambda \Psi_i + C_i = \lambda(\sum_{f_l \in M_i} \mu^l) + C_i = P_i$ and $\mu^k =$

$\frac{s_i^k - c_i^k}{\lambda}$ for every $i \in G^k$ and $f_k \in \mathcal{F}$. As function τ^k is strictly increasing in μ^k on $(1, \infty)$ for any $f_k \in \mathcal{F}$, equations (8a-8d) determine each firm's optimal η-markup when the value of H and the η-markups of other firms providing complementary products are given. Inspecting equations (8a) and (8b), we see why the standard aggregative games approach does not work: No common aggregator can fully summarize the effects of all firms' η-markups on the solution in μ^1 and μ^2. For example, μ^3 affects the solution in μ^1 by changing the output of function h_2 and the aggregator H, while μ^3 only affects the solution in μ^2 through the aggregator H. Hence, a firm's own first-order conditions alone cannot determine the standard fitting-in function.

To solve this problem, we use the smallest system of equations to identify the fitting-in function for every firm in $\mathcal{F}$. For any given value of $H \geq H^0$, equations (8a-8c) for firms in L_1 should all be used to determine a unique solution in $(\mu^k)_{f_k \in L_1}$ by applying the Poincaré-Hopf index theorem, and this solution is denoted by $(g^k(H))_{f_k \in L_1}$. Similarly, we can obtain $g^4(H)$ from equation (8d) for firm f_4 in L_2. Firms' fitting-in functions also determine final products' fitting-in functions $(m_i(H))_{i \in \mathcal{N}}$, where $m_i(H) = \sum_{f_l \in M_i} g^l(H)$ for every $i \in \mathcal{N}$. All we need to do now is use equation (8e) to find an H such that

$$H = h_1(t_1(m_1(H))) + h_2(t_2(m_2(H))) + h_3(t_3(m_3(H))) + H^0,$$

which is a more tractable uni-dimensional fixed-point problem.

We note that this new method combines the standard aggregative games approach and the traditional method for analyzing the equilibrium existence problem. When the standard aggregative games approach is not applicable, it seems possible to directly evaluate the solution in $(\mu^k)_{f_k \in \mathcal{F}}$ of equations (8a-8e) for all firms in $\mathcal{F}$ as in the study by Konovalov and Sándor (2010). However, this approach is difficult to apply in a general network, and the fact that firms' η-markups can be either strategic substitutes or strategic complements further complicates the problem and leads to limited applications. By using the common aggregator H, firms' η-markups and final products' aggregate markups within a separable component are easier to analyze. We can then simplify the equilibrium existence problem, similar to a standard aggregative game (Jensen (2018), Nocke and Schutz (2018)). As we will see later on, the partially aggregative games approach is also helpful in providing comparative statics, obtaining equilibrium uniqueness, and addressing important policy questions.

4.2 Equilibrium Existence and Characterization

To complete the proof of the existence of a pricing equilibrium, we need the following key assumptions:

Assumption 1. *For every $i \in \mathcal{N}$, $\eta_i'(P_i) \geq 0$ whenever $\eta_i(P_i) > 1$.*

Assumption 2a. *For every $i \in \mathcal{N}$, $\lim_{P_i \to \infty} \eta_i(P_i) > |M_i|$, and $H^0 > \underline{H}^0 \geq 0$.*[23]

Assumption 2b. *For every $i \in \mathcal{N}$, $\lim_{P_i \to \infty} \eta_i(P_i) = \infty$.*

Theorem 1. *Given any network $(\mathcal{F}, \mathcal{N})$, suppose that the demand system $((h_i)_{i \in \mathcal{N}}, H^0)$ satisfies Assumption 1. There exists a pricing equilibrium if either Assumption 2a or Assumption 2b holds.*

Assumption 1, which is sometimes called Marshall's second law of demand, states that the price elasticity of the monopolistic competition demand for each final product $i \in \mathcal{N}$ should be nondecreasing in its own price (Nocke and Schutz (2018)). A similar assumption that each final product's own-price elasticity of demand is nondecreasing in its own price is often used in oligopoly models (Hendricks et al. (1997), Vives (1999), Quint (2014a)). Since prices are finite, Assumption 2a is added to ensure the existence of a pricing equilibrium. Conditions similar to Assumption 2a have also been used in the literature for the same reason.[24] Summing the inverse elasticity rules $\frac{s_i^k - c_i^k}{P_i} = \frac{\mu^k}{\eta_i(P_i)}$ of the firms supplying final product i yields $\frac{P_i - C_i}{P_i} = \frac{\Psi_i}{\eta_i(P_i)}$, so the aggregate markup must stay below $\eta_i(P_i)$ for a finite P_i. Since $\Psi_i \geq |M_i|$, the limit of $\eta_i(P_i)$ as $P_i \to \infty$ (i.e., an upper bound of $\eta_i(P_i)$ under Assumption 1) should be higher than the number of firms supplying final product i. In markets without complements, $|M_i| = 1$ for every $i \in \mathcal{N}$ and this condition is automatically satisfied (Nocke and Schutz (2018)). Moreover, the value of H^0 should be higher than a lower bound $\underline{H}^0$ determined by the network $(\mathcal{F}, \mathcal{N})$, the total marginal costs $(C_i)_{i \in \mathcal{N}}$, and the functions $(h_i)_{i \in \mathcal{N}}$. The reason is that a multiproduct firm chooses a common η-markup, while this firm's final products may differ in the total number of suppliers and in the upper bound on $\eta_i(P_i)$: The η-markup that is optimal for the firm as a whole may push some aggregate markups of its final products to their limits, leading to infinite prices. Sufficient competition from the outside option lowers all η-markups and rules this out.[25] We

[23] The lower bound $\underline{H}^0$ is formally defined in the proof of Theorem 1 in the Appendix. We illustrate the key steps to identify a lower bound here. Inspecting equation (6) with $\gamma_i' < 0$ and $t_i' > 0$ for any $\Psi_i < \lim_{P_i \to \infty} \eta_i(P_i)$ and $i \in \mathcal{N}$ under Assumption 1, we see that, by setting $\mu^l = 1$ for every f_l with $G^l \cap G^k \neq \varnothing$ and $f_l \neq f_k$, the corresponding μ^k that solves equation (6), which is strictly decreasing in $H \in (0, \infty)$, is an upper bound of μ^k with a given $H > 0$. Then, we can pick the value of H, under which the corresponding maximum aggregate markup of any final product $i \in \mathcal{N}$ is less than $\lim_{P_i \to \infty} \eta_i(P_i)$, as the lower bound $\underline{H}^0$. Therefore, if $H^0 > \underline{H}^0$, each firm's optimal choice of η-markup under a given $H \geq H^0$ and the pricing equilibrium must correspond to finite prices.

[24] In the pricing of pure complementary products with an iso-elastic demand, a similar assumption on the own-price elasticity of demand needs to be imposed to ensure the existence of a pricing equilibrium (e.g., Hendricks et al. (1997)).

[25] Our key step in identifying firms' fitting-in functions based on the Poincaré-Hopf index theorem requires the right-hand side of equation (6) to be continuously differentiable at every solution, which may not hold at points with infinite prices.

identify the bound $\underline{H}^0$ that is sufficient for any network and functions $(h_i)_{i\in\mathcal{N}}$, and other possible $\underline{H}^0$ can be used when the network is specified.[26]

Assumption 2b, which can be considered a special case of Assumption 2a with $\underline{H}^0 = 0$, ensures the existence of a pricing equilibrium with any positive value of H^0. In other words, there exists a pricing equilibrium for any $(\mathcal{F},\mathcal{N})$ and total marginal costs $(C_i)_{i\in\mathcal{N}}$ if, instead, Assumption 2b holds. Intuitively, an unbounded elasticity measure (i.e., $\eta_i(P_i)$) ensures that $\frac{P_i-C_i}{P_i} = \frac{\Psi_i}{\eta_i(P_i)}$ always has a finite solution in P_i for any $\Psi_i > 1$. MNL demand satisfies Assumptions 1 and 2b, which means that this frequently used demand continues to have a pricing equilibrium in this more general setting. Unlike MNL, CES demand satisfies Assumption 1 but not Assumption 2b. If there are changes in the composition and ownership structure of the products $(\mathcal{F},\mathcal{N})$, the marginal costs $(C_i)_{i\in\mathcal{N}}$, the functions $(h_i)_{i\in\mathcal{N}}$, or some combination thereof, the value of H^0 or the limit of $\eta_i(P_i)$ for $i \in \mathcal{N}$ may need to be adjusted to satisfy Assumption 2a and ensure the existence of a pricing equilibrium.

Finally, we show that the equilibria of the pricing game can be effectively characterized by the partially aggregative games approach.

Proposition 1. *Under Assumptions 1 and 2a, the set of equilibrium aggregators is the set of fixed points of the aggregate fitting-in function* Γ*, given in equation (7). Suppose that* H^* *is an equilibrium aggregator. Then, in the equilibrium with* H^**,*

(*a*) *the price of final product* $i \in \mathcal{N}$ *is* $t_i(m_i(H^*))$*,*

(*b*) *the profit of firm* $f_k \in \mathcal{F}$ *is* $g^k(H^*) - 1$*, and*

(*c*) *consumer surplus is given by* $\log(H^*)$*.*

According to equations (5) and (6), in equilibrium, prices of all final products, profits of all firms, and consumer surplus only rely on the total costs of final products, $(C_i)_{i\in\mathcal{N}}$, not on the individual costs of intermediate products, $(c_i^k)_{f_k\in\mathcal{F},i\in G^k}$. The equilibrium price of an intermediate product, which depends on the cost of this intermediate product, can be determined by plugging the firm's equilibrium profit and the final product's equilibrium price into equation (4). The pricing function t_i maintains the same property as in Nocke and Schutz (2018): Under Assumption 1, t_i is differentiable and strictly increasing on $(1,\bar{\mu}_i)$ for any $i \in \mathcal{N}$, where $\bar{\mu}_i = \lim_{P_i\to\infty}\eta_i(P_i)$. This means that the price of a final product increases when its aggregate markup increases. However, the properties of the fitting-in functions $(m_i)_{i\in\mathcal{N}}$ and $(g^k)_{f_k\in\mathcal{F}}$ depend on the given network $(\mathcal{F},\mathcal{N})$, and we will study them in Section 5. In the following sections (except Example 2 in Section 7), Assumptions 1 and 2a are imposed on any given network so that the fitting-in functions are well-defined.

[26] Under CES demand, $\underline{H}^0 < \max_{f_k\in\mathcal{F}} \frac{\sigma-1}{\sigma-\max_{i\in G^k}|M_i|}\sum_{j\in G^k} h_j(t_j(|M_j|))$, where $t_j(|M_j|)$ is the price of final product j under monopolistic competition with $\mu^k = 1$ for any $f_k \in \mathcal{F}$ and hence $\Psi_i = |M_i|$ for any $i \in \mathcal{N}$. Therefore, Assumption 2a does not require a large outside option. $\underline{H}^0$ is less than a multiple of $\sum_{j\in G^k} h_j(t_j(|M_j|))$ for a single firm f_k, rather than of all final products in the network.

5 Profits, Prices, and Competition Level

The value of the aggregator H provides a measure of the overall competition level in a market. A higher H corresponds to a higher level of competition with a higher consumer surplus. The value of the aggregator H also has a direct relationship with the entry of a new competitor, as discussed later in Section 6.1. According to Theorem 1 and Proposition 1, the fitting-in function g^k for firm $f_k \in \mathcal{F}$ indicates how firm f_k will choose its η-markup, which also determines f_k's profit, in response to the competition level, and the fitting-in function m_i for final product $i \in \mathcal{N}$ tells us how final product i's total price responds to the competition level. In this section, we investigate how these fitting-in functions depend on the given network of complements and substitutes, which will also be essential for interpreting the applications in Section 6.

5.1 Network Topology

We classify economies with complements and substitutes based on the relationships between firms with complementary intermediates. There are two major types of networks:

(**S**) Symmetric network: For any two firms, f_k and f_l, if $G^k \cap G^l \neq \varnothing$, then $G^k = G^l$.[27]

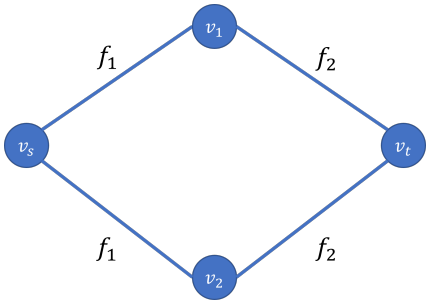
(**A**) Asymmetric network: There exist two firms, f_k and f_l, such that $G^k \cap G^l \neq \varnothing$ and $G^k \neq G^l$.

For any two firms f_k and f_l, if $G^k \cap G^l \neq \varnothing$, then these two firms provide complementary products for final products in $G^k \cap G^l$. Under an (**S**) network, any two firms producing complements are "loyal" to each other in the sense that neither of the two firms works with other firms (or by itself) to introduce an additional substitutable final product. The network of complements and substitutes in the baseline model of Quint (2014a) is a special case of (**S**) networks. However, as in the market structures studied by Salinger (1991) and Luco and Marshall (2020), firms often produce intermediate products for different firms in real-world markets, and we introduce asymmetric networks to capture more general relationships between firms' products. Under an (**A**) network, there must exist at least two firms with complements, such that the firms' intermediates are for two different groups of final products. Moreover, we distinguish two types of (**A**) networks:

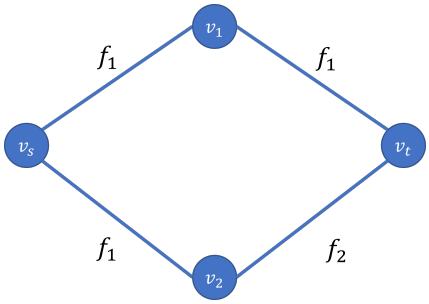
(**A1**) One-sided asymmetric network: For any two firms, f_k and f_l, if $G^k \cap G^l \neq \varnothing$, then either $G^k \subseteq G^l$ or $G^l \subseteq G^k$.

(**A2**) Two-sided asymmetric network: There exist two firms, f_k and f_l, such that $G^k \cap G^l \neq \varnothing$, $G^k \nsubseteq G^l$, and $G^l \nsubseteq G^k$.

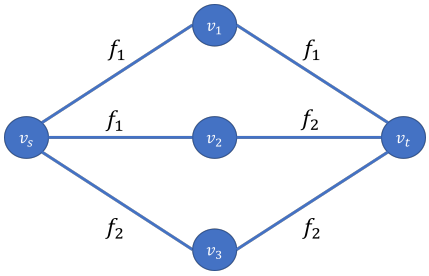
[27] The market without any complements, as shown in Nocke and Schutz (2018), is considered a special case of the symmetric network.



(a) Symmetric Network



(b) One-sided Asymmetric Network



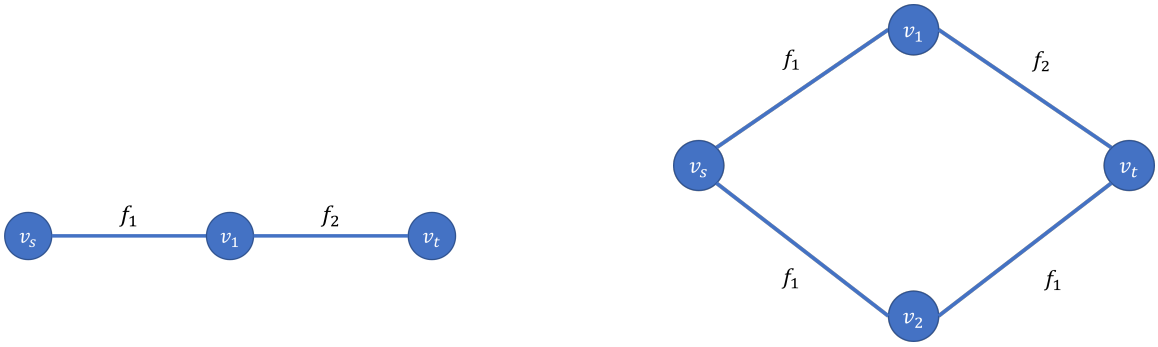
(c) Two-sided Asymmetric Network

Figure 4: An illustration of the three types of networks.

In an $(\mathbf{A1})$ network, if two firms produce some complements, then one firm's products are complementary to a subset of the other firm's products, meaning that at most one firm participates in a competing final product that excludes the other. This type of network generalizes the relationships between firms' products in Salinger (1991) and Luco and Marshall (2020) by relaxing restrictions on the number of firms and final products and accommodating additional competitors such as other separable components. However, Reisinger et al. (2021) demonstrate that firm f_k, where $G^k \subseteq G^l$ and $G^k \neq G^l$, may decide to intensify competition against firm f_l by introducing an additional competing product. Such a new product can reduce the double marginalization problem and improve f_k's profit. This strategic behavior results in an $(\mathbf{A2})$ network, where both firms producing complementary products may work with third parties or operate independently to introduce additional competing final products. Figure 4 illustrates these network topologies. Finally, we characterize the relationship between a network $(\mathcal{F}, \mathcal{N})$ and its separable components as follows: A network is $(\mathbf{S})$ if and only if every separable component is $(\mathbf{S})$; a network is $(\mathbf{A1})$ if and only if every separable component is either $(\mathbf{S})$ or $(\mathbf{A1})$ with at least one separable component being $(\mathbf{A1})$; a network is $(\mathbf{A2})$ if and only if at least one separable component is $(\mathbf{A2})$.

5.2 Firms' Fitting-in Functions

Since a firm's fitting-in function is determined by equations (6) for firms within the same separable component, we provide a sufficient condition on the separable component under which the firm-level fitting-in functions are strictly decreasing.



(a) Firms f_1 and f_2 form an $(\mathbf{S})$ separable component.

(b) Firms f_1 and f_2 form an $(\mathbf{A1})$ separable component.

Figure 5: A comparison of symmetric and one-sided asymmetric network.

Proposition 2. *The firm-level fitting-in function $g^k(H)$ for any $f_k \in L_\alpha$ is differentiable and strictly decreasing on (H^0, ∞) when S_α is an $(\mathbf{S})$ network. Moreover, $\lim_{H\to\infty} g^k(H) = 1$.*

Proposition 2 shows that firm f_k reduces its η-markup under any $(\mathbf{S})$ network as competition intensifies. Firm f_k and its complementary firms choose identical η-markups in this case by equation (6). Consequently, intensified competition (reflected by a higher H) can only stimulate the firms to reduce these η-markups. However, if the network becomes asymmetric, firms' η-markups may not be strictly decreasing when competition intensifies. Therefore, unlike markets with only substitutes, increased competition in markets with asymmetric networks may actually benefit some firms. To understand this result, we first note that with a given aggregator $H > H^0$, equation (6) determines each firm's optimal choice of the η-markup as a best-response to the η-markups of its complementary firms:

Lemma 1. *For any two firms $f_k, f_l \in \mathcal{F}$, if $G^k \cap G^l \neq \varnothing$, then firm f_k's optimal choice of μ^k is strictly decreasing in μ^l for any given $H > H^0$.*

The fact that firms' η-markups are strategic substitutes complicates the properties of the fitting-in functions with asymmetric networks. The following lemma further characterizes the fitting-in function for the firm that supplies every final product in an $(\mathbf{A1})$ separable component:

Lemma 2. *If firm f_k produces intermediate products for all final products in an $(\mathbf{A1})$ separable component S_α, then $g^k(H)$ is differentiable and strictly decreasing on (H^0, ∞).*

If the separable component S_α is $(\mathbf{A1})$, then there always exists such a firm $f_k \in L_\alpha$, which is involved in all final products in S_α (i.e., $G^k = S_\alpha$). Since competition directly affects all f_k's products, f_k reduces its η-markup and earns a lower profit as H increases.

Figure 5 illustrates how such increases in η-markups occur. In Figure 5a, firm f_1 and firm f_2 form an $(\mathbf{S})$ network within a separable component. While an increase in

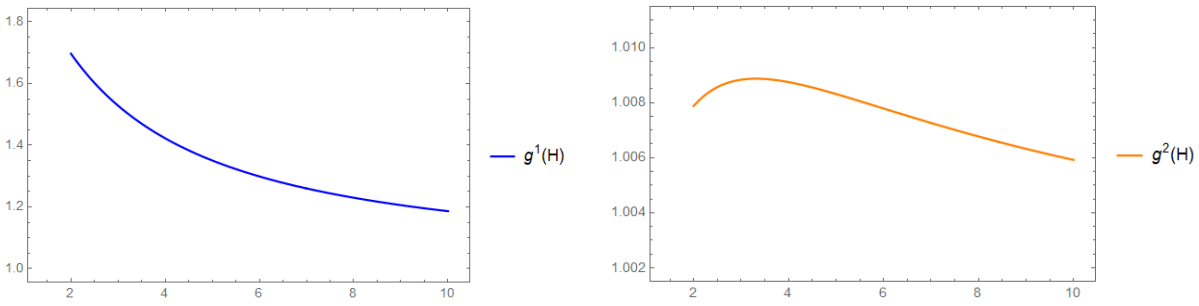


Figure 6: Fitting-in functions for firms in Figure 5b.

H exerts downward pressure on firm f_1's η-markup, this effect is partially offset by the simultaneous reduction in firm f_2's η-markup, reflecting the strategic substitutability of their η-markups.

In contrast, Figure 5b illustrates an (**A1**) network where firm f_1 introduces an additional competing product within the same separable component. Because firm f_1 produces the second product exclusively, as in standard horizontal competition, it competes more intensively with other separable components. Consequently, as H increases, f_1 may reduce its η-markup more substantially to enhance the competitiveness of the second product, and its choice of η-markup becomes less sensitive to a change in f_2's η-markup. This more aggressive reduction in f_1's η-markup can dominate the direct η-markup reduction effect on f_2 from a higher H, and by strategic substitutability, f_2 raises its own η-markup. As illustrated in Figure 6 under a CES demand specification, the fitting-in function g^2 is not strictly decreasing in H.[28] Similarly, an increase in H may raise a firm's η-markup within (**A2**) separable components.

Lastly, we note that such potential increases in η-markups in asymmetric separable components cannot be sustained when the increase in H is sufficiently large. The following proposition demonstrates that, if competition significantly intensifies, all firms will eventually reduce their η-markups under any network. A reduction in one firm's η-markup can only raise the η-markups of some other firm when the increase in H remains within a limited range.

Proposition 3. *With a given aggregator $H > H^0$, there exists $\underline{H} > H$ such that $g^k(H) > g^k(\hat{H})$ for every $f_k \in L_\alpha$ and every $\hat{H} > \underline{H}$ when S_α is an (**A1**) or (**A2**) network. Moreover, $\lim_{H\to\infty} g^k(H) = 1$.*

5.3 Final Products' Fitting-in Functions

We now evaluate the fitting-in function m_i for final product $i \in S_\alpha$ and $S_\alpha \in S$. Following equation (5), we know that $m_i = \sum_{f_l \in M_i} g^l$.

[28] The specifics of the CES demand are the following: $h_1(P_1) = (P_1)^{-2.4}$, $h_2(P_2) = 6(P_2)^{-2.4}$, and $C_1 = C_2 = 1$. Also, $H \in [2, 10]$.

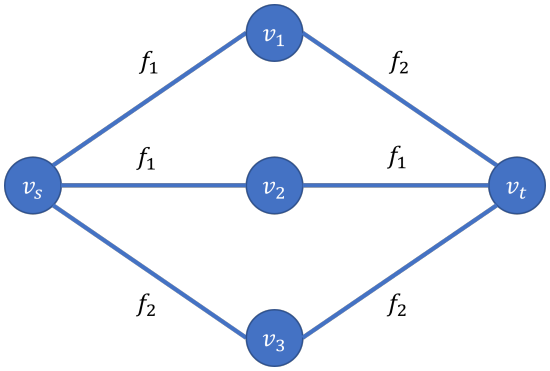


Figure 7: Firms f_1 and f_2 form an (**A2**) separable component.

Proposition 4. *The fitting-in function $m_i(H)$ for any $i \in S_\alpha$ is differentiable and strictly decreasing on (H^0, ∞) when S_α is an (**S**) or (**A1**) network. Moreover, $\lim_{H\to\infty} m_i(H) = |M_i|$.*

When competition intensifies, the aggregate markups of all final products in any (**S**) or (**A1**) separable component decrease, thereby reducing the prices of the final products as the pricing function t_i is strictly increasing. As shown in Section 5.2, some firms' η-markups may still get higher in an (**A1**) separable component. In contrast, m_i may not be strictly decreasing in an (**A2**) separable component. We can revise Figure 5b and turn it into an (**A2**) network in Figure 7, where firm f_2 also offers a competing final product by itself (i.e., final product 3). Now, if competition intensifies, the reduction in firm f_1's η-markup may stimulate firm f_2 to raise its η-markup, making the aggregate markup of final product 3 higher.

Finally, based on Proposition 3, if H significantly increases, all aggregate markups will go down in an (**A2**) separable component.

Corollary 1. *With a given aggregator $H > H^0$, there exists $\underline{H} > H$ such that $m_i(H) > m_i(\hat{H})$ for every $i \in S_\alpha$ and every $\hat{H} > \underline{H}$ when S_α is an (**A2**) network. Furthermore, $\lim_{H\to\infty} m_i(H) = |M_i|$.*

6 Comparative Statics

Similar to a standard aggregative game illustrated in Jensen (2018) and Nocke and Schutz (2018), the partially aggregative games approach can be used to perform comparative statics on the set of equilibria. To see how a change in a parameter may affect the equilibrium outcomes, we first need to study how this change affects the pricing functions and the fitting-in functions for final products and firms. Second, we determine the impact on the aggregate fitting-in function and analyze how the set of equilibrium aggregators is changed. Finally, we use the pricing functions and fitting-in functions for final products and firms to determine changes in the key variables.

6.1 Outside Option/Entry Analysis

We first study the effects of an increase in the value of the outside option H^0. This analysis also illustrates how the entry of a new firm providing substitutable final products for existing products affects the equilibrium outcomes. In this case, changes in the parameter H^0 only affect the domain of the fitting-in functions. Then, an increase in H^0 (or the entry of a new firm providing substitutes) simply shifts the aggregate fitting-in function up and raises the lowest and highest equilibrium aggregator levels. Therefore, we can show that an increase in H^0 improves the lowest and highest levels of consumer surplus in equilibrium, and we can apply our previous results about the properties of the pricing and fitting-in functions for final products and firms to determine the effects on prices and profits in these two extreme equilibria.

Proposition 5. *Suppose that H^0 increases or a new firm providing substitutable final products in a new separable component enters. In the equilibria with the lowest and highest aggregator H, this induces:*

(a) an increase in consumer surplus,

(b) decreases in the profits of all firms in every $(\mathbf{S})$ separable component,

(c) a decrease in the profit of a firm f_k with $G^k = S_\alpha$ in an $(\mathbf{A1})$ separable component S_α, and

(d) decreases in the prices of all final products in every $(\mathbf{S})$ or $(\mathbf{A1})$ separable component.

Proposition 5 identifies the separable components of a given network where an increase in H^0 reduces firms' profits or prices of the final products in equilibrium. As we have shown in Section 5, the firm f_k with $G^k = S_\alpha$ in an $(\mathbf{A1})$ separable component offers a stimulus for other firms in this separable component to raise their η-markups, and we can easily derive numerical examples where an increase in H^0 can improve the profits of some other firms providing complements for f_k in an $(\mathbf{A1})$ separable component. Moreover, in an $(\mathbf{A2})$ separable component, the prices of some final products may also get higher after an increase in H^0. Therefore, unlike models with pure substitutes, as shown in Nocke and Schutz (2018), or with single-product firms, as shown in Quint (2014a), a new competitor can benefit incumbent firms and raise the prices of existing final products in an asymmetric network.

Proposition 3 also implies the following corollary, which establishes that if the increase in the value of the outside option, H^0, is sufficiently large, or equivalently, if the new firm is competitive enough, the profits of all incumbent firms and the prices of all existing final products will go down.

Corollary 2. *Given an H^0 and network $(\mathcal{F}, \mathcal{N})$, there exists $\hat{H}^0 > H^0$ such that if the value of the outside option increases to a level higher than $\hat{H}^0$, the profits of all firms and the prices of all final products decrease in the equilibria with the lowest and the highest aggregator H.*

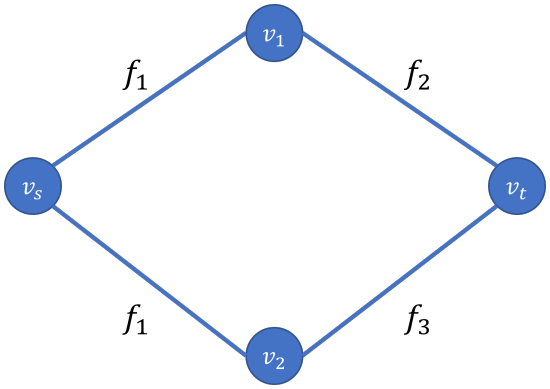


Figure 8: The (**A1**) network for Example 1.

6.2 Marginal Cost

As discussed in Nocke and Schutz (2018), a cost reduction (or similarly, a quality improvement) in one final product can lead to ambiguous effects on consumer surplus in markets where multiproduct firms provide pure substitutes. One intuition for this result follows from Edgeworth's paradox of taxation (Edgeworth (1925), Salinger (1991), Luco and Marshall (2020), Armstrong and Vickers (2023)). If a multiproduct firm has an exogenous cost reduction in one of its products (maybe caused by a tax reduction), it has incentives to raise other products' prices to divert sales toward that product with a lower marginal cost. As a result, a cost reduction may raise multiple products' prices and reduce the price of the final product with a lower marginal cost. The overall impact of this cost reduction on consumer surplus also becomes uncertain under the general demand system $((h_i)_{i\in\mathcal{N}}, H^0)$.

In this section, we show that the impact of a cost reduction on consumers also depends on the network of complements and substitutes by focusing on CES or MNL demands. Since a merger of complements can be similarly considered a reduction in marginal cost, as shown in Salinger (1991), this analysis also provides implications about the competitive effects of a merger of complements between two firms, f_k and f_l, where either $G^k \subseteq G^l$ or $G^l \subseteq G^k$.[29] In this section, H^* denotes the equilibrium aggregator before the cost reduction, and $m_i(H^*)$ and $g^k(H^*)$ are the corresponding equilibrium aggregate markup of final product $i \in \mathcal{N}$ and η-markup of firm $f_k \in \mathcal{F}$.

We first study an example with an (**A1**) network, which is illustrated in Figure 8. This network is the focus of the empirical study by Luco and Marshall (2020).[30] It has

[29] Consider two hypothetical single-product firms, f_k and f_l, providing intermediates for final product $i \in \mathcal{N}$. Firm f_k's profit maximization problem is choosing s_i^k to maximize $(s_i^k - c_i^k)\hat{D}_i(s_i^k + s_i^l, H(P))$, which is equivalent to choosing P_i to maximize $(P_i - (c_i^k + s_i^l))\hat{D}_i(P_i, H(P))$ by a change of variables. If f_k and f_l merge and only f_k is left, f_k's new profit maximization problem becomes choosing s_i^k to maximize $(s_i^k - (c_i^k + c_i^l))\hat{D}_i(s_i^k, H(P))$. Since $s_i^l > c_i^l$ before the merger, this merger of complements simply reduces the marginal cost of final product i for the remaining firm. The same change of variables applies to mergers of multiproduct firms whenever $G^k \subseteq G^l$ or $G^l \subseteq G^k$.

[30] The analysis in Example 1 is slightly different from the study in Luco and Marshall (2020) as firm

only one separable component, which helps us to see the price changes more clearly.

Example 1. *Suppose that the given network only has one separable component depicted in Figure 8. The MNL demand for each final product $i \in \{1,2\}$ satisfies $h_i(P_i) = \exp(a_i - P_i)$, where $a_i \in \mathbb{R}$. A decrease in C_1 reduces the equilibrium price of final product 1. Moreover, this cost reduction increases the equilibrium price of final product 2 if one of the following conditions holds:*

(a) $a_1 = a_2$ and $C_1 = C_2$.

(b) $H^0 > h_2(m_2(H^*) + C_2) = h_2(t_2(m_2(H^*)))$.

Even though the total price of final product 1 is always lower after the cost reduction, the price change in final product 2 is uncertain. Firm f_1 tries to raise its price for final product 2 so that more consumers choose final product 1, which becomes more profitable after the cost reduction. However, firm f_3 also tries to reduce the price of its intermediates for final product 2 so that it can better compete against final product 1. Example 1 provides some sufficient conditions under which the total price of final product 2 increases. Therefore, the cost reduction can easily lead to both price increases and price decreases in a network of complements and substitutes. However, in the following proposition, our partially aggregative games approach allows us to develop a general result that the overall impact of a cost reduction on consumers is always positive under any $(\mathbf{S})$ or $(\mathbf{A1})$ network.

Proposition 6. *Whenever the given network is $(\mathbf{S})$ or $(\mathbf{A1})$, a reduction in C_i for final product $i \in \mathcal{N}$ improves equilibrium consumer surplus with CES or MNL demands.*

When markets only have pure substitutes (i.e., a special $(\mathbf{S})$ network) under CES or MNL demands, Nocke and Schutz (2018) have shown that a cost reduction in one of a firm's products can only increase consumer surplus, even though this firm's other final products, whose marginal costs are unchanged, become more expensive. Proposition 6 indicates that when multiproduct firms provide complements and substitutes, this important result still holds for any $(\mathbf{S})$ or $(\mathbf{A1})$ network. Hence, if a cost reduction reduces consumer surplus, the network must be an $(\mathbf{A2})$ network. The next result illustrates how a cost reduction can reduce consumer surplus in an $(\mathbf{A2})$ network.

Proposition 7. *Consider an $(\mathbf{A2})$ network with only one separable component, depicted in Figure 7. The MNL demand for each final product is given by $h_1(P) = h_2(P) = h_3(P) = h(P) = \exp(a - P)$, where $a \in \mathbb{R}$. Also, $C_1 = C_2 = C_3 = C > 0$.[31] Then, a decrease in C_1 by $\Delta C_1 > 0$ induces:*

f_2 chooses a new price after the cost reduction. However, as shown in the proof of Example 1, firm f_2's strategic responses do not play a role, and this result still holds when f_1 and f_2 merge.

[31] Proposition 8, a uniqueness result independent of this section, shows that the pricing equilibrium with symmetric η-markups is the unique pricing equilibrium in this network.

(a) increases in the equilibrium prices of final products 2 and 3,

(b) a decrease in equilibrium consumer surplus when $H^0 < \min(h(m_1(H^*)+C), \tilde{H}^0)$, *where* $\tilde{H}^0 = h(g(H^*)+C)(\frac{1}{\Delta C_1} - \frac{1+\Delta C_1}{\Delta C_1 \exp(g(H^*)\Delta C_1)})$ *and* $g^1(H^*) = g^2(H^*) = g(H^*)$.[32]

(c) an increase in the equilibrium price of final product 1 if and only if $H^0 < \bar{H}^0$, *where* $\bar{H}^0 \equiv h(g(H^*)+C)(\frac{2(g(H^*)-2)(1-\exp(\frac{-\Delta C_1}{2}))}{\Delta C_1} - \exp(\frac{-\Delta C_1}{2}))$, *and* $\bar{H}^0 < \min(h(m_1(H^*)+C), \tilde{H}^0)$.

The assumption that h_i and C_i are the same for each final product ensures that the network is not leaning towards a one-sided asymmetric network or a market with pure substitutes. In Figure 7, if the total marginal cost of providing final product 1 decreases, both firm f_1 and firm f_2 have incentives to raise their prices of final products 2 and 3 to divert consumers toward final product 1. These price increases in final products 2 and 3 reinforce each other. Firm f_1 has less concern that its price increase in final product 2 may drive consumers to choose final product 3 if firm f_2 also raises the price of final product 3. Proposition 7 (b) also indicates that the value of the outside option (i.e., H^0) should be sufficiently low for the cost reduction to reduce consumer surplus. This condition has two parts. First, $H^0 < h(m_1(H^*)+C) = h(t_1(m_1(H^*)))$ means that the outside option is less attractive to consumers than final product 1 before the cost reduction. Dividing both sides by H^*, this is an easy-to-verify condition that the choice probability of the outside option (or equivalently, the fraction of consumers choosing it, as in Quint (2014b)) is smaller than that of final product 1 before the cost reduction. Under the MNL demand, it also implies that the diversion ratio from final product 2 (or 3) to final product 1 is higher than the diversion ratio from final product 2 (or 3) to the outside option. Hence, when C_1 drops, firms are less concerned that increases in the prices of final products 2 and 3 may drive consumers to choose the outside option. Second, as $\tilde{H}^0$ is strictly decreasing in ΔC_1 when $H^0 < h(m_1(H^*)+C)$, $H^0 < \tilde{H}^0$ indicates that the value of the outside option may need to be lower when the cost reduction is more substantial. By reducing H^0, the two firms can more substantially raise their prices to counteract the potential positive effects on consumer surplus from the cost reduction (i.e., price reduction in final product 1). Proposition 7 (c) then shows a condition stricter than Proposition 7 (b) under which a cost reduction can even raise all final products' prices. In this case, the price increases in final products 2 and 3, which equal the two firms' η-markup increases, together exceed the cost reduction in final product 1, so the price of final product 1 also increases. This shows that Edgeworth's paradox can arise in a market where two multiproduct firms produce both complements and substitutes.

We can modify Figure 7 and turn it into Figure 9, which further explains why the

[32] Changes in H^0 also affect the value of H^*. However, as shown in our proof, $2 < g(H^*) < 2.13$ in this network with $H^0 < \min(h(m_1(H^*)+C), \tilde{H}^0)$, which implies that by reducing the value of H^0, the condition on H^0 can be satisfied.

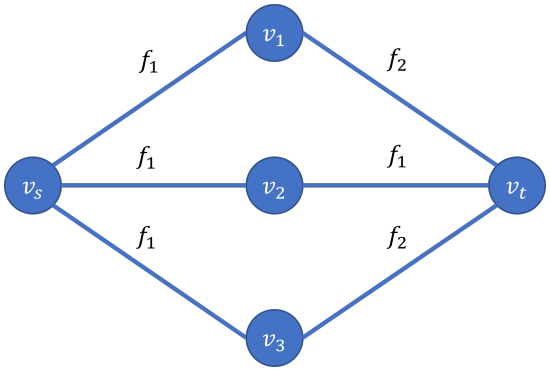


Figure 9: Firms f_1 and f_2 form an (**A1**) separable component.

impact of a cost reduction on consumers under an (**A2**) network differs from that under an (**A1**) network. In this (**A1**) network depicted in Figure 9, firm f_2's intermediate products are complementary to a subset of firm f_1's intermediate products. A cost reduction in final product 1 again stimulates firms f_1 and f_2 to increase their prices to divert consumers toward final product 1. However, the price increase in firm f_2's intermediate product for final product 3 now hurts firm f_1's profit, which, from firm f_1's perspective, can be considered an increase in marginal cost in selling final product 3. Hence, this price increase may actually reduce firm f_1's incentive to raise the price of final product 2 as final product 3 becomes less profitable to sell. With less significant price increases in final products 2 and 3, such a cost reduction in final product 1 always improves equilibrium consumer surplus, as shown in Proposition 6.

Finally, we note that the analysis in Proposition 7 can be applied to a merger of complements like in other types of networks. If we add firm f_3 to final product 1 of Figure 7 and consider a merger between f_1 and f_3, this merger can be similarly considered a cost reduction in final product 1.[33] Hence, this merger can reduce consumer surplus when the value of H^0 is small enough. However, the (**A2**) network also gives rise to a new type of merger of complements. If we consider the merger between firms f_1 and f_2 in Figure 7, the competition between f_1's final product 2 and f_2's final product 3 resembles the standard horizontal competition, and this merger can no longer be considered a reduction in marginal cost. We study this type of merger in the international airline context in our companion paper (Tan and Zhang (2025)).

[33]Since $1 < g^3(H^*) < 2$ at the pre-merger equilibrium aggregator H^* under the assumptions in Proposition 7, adding firm f_3 increases the cost of final product 1 by less than two. Proposition 7 remains valid for any $C_1 \geq C$, with $h(m_1(H^*) + C)$ replaced by $h(t_1(m_1(H^*)))$, because its proof only uses $C_2 = C_3$ and $C_1 \geq C$. Applying it with $C_1 = C + g^3(H^*)$ and $\Delta C_1 = g^3(H^*)$, the merger reduces consumer surplus when $H^0 < \min(h(t_1(m_1(H^*))), \tilde{H}^0)$, and it cannot raise the price of final product 1 because $\Delta C_1 < 2$ and $g(H^*) < 2.13$ imply $\bar{H}^0 < 0$. The merger can raise all final products' prices only if it generates substantial additional synergies that further reduce C_1 (i.e., enlarge ΔC_1).

7 Equilibrium Uniqueness

In this section, we study the uniqueness of a pricing equilibrium. With our partially aggregative games approach, the uniqueness of a pricing equilibrium can be shown by evaluating whether the aggregate fitting-in function, $\Gamma(H)$, has a unique fixed point or not. We focus on studying the uniqueness of a pricing equilibrium with MNL or CES demands. Proposition 8 shows a general result under MNL demand:

Proposition 8. *Under any network, the pricing equilibrium is unique with MNL demand.*

In the case of CES demand, when the network is not (**A2**), by using Proposition 2 and Lemma 2, we can show that the pricing equilibrium is always unique. In the case of an (**A2**) network, we show that when the number of firms or the number of final products is small within any (**A2**) separable component, the pricing equilibrium is unique. No restriction is imposed on an (**S**) or (**A1**) separable component of an (**A2**) network. In summary, we obtain the following proposition:

Proposition 9. *The pricing equilibrium is unique under CES demand if, for every $S_\alpha \in S$, one of the following conditions holds:*

*(a) S_α is (**S**) or (**A1**).*

*(b) S_α is (**A2**) and $|L_\alpha| \leq 3$.*

*(c) S_α is (**A2**), $|S_\alpha| \leq 3$, and $G^k \neq G^l$ for any $f_k, f_l \in L_\alpha$.*

Even though Proposition 9 is not as general as the case with MNL demand, it is sufficient to capture the major categories of networks in this paper. Also, in many applications, strategic responses from two or three firms with a small number of final products are the focus of the study, where other firms are often incorporated into the outside option (e.g., the study of a vertical merger in Luco and Marshall (2020) or the oligopoly model in Reisinger et al. (2021)). CES demand still leads to a unique pricing equilibrium in these cases. By adding more restrictions to the network structure, our approach can be used to show the uniqueness of a pricing equilibrium under CES demand with more products or firms (for example, the locations of the fourth and fifth firms are specified based on the requirements of the application).

Example 2. *There exist an (**A2**) network, total marginal costs $(C_i)_{i\in\mathcal{N}}$, and a CES demand system with $H^0 < \underline{H}^0$ that have multiple pricing equilibria.*

Unlike markets with pure substitutes in Nocke and Schutz (2018), the pricing equilibrium need not be unique under CES demand in markets with complements and substitutes. In the Online Appendix, we present an example in an (**A2**) network with multiple pricing equilibria. Although the pricing equilibria include only finite prices, a key feature of this example is that $H^0 < \underline{H}^0$, under which firms' optimal prices with

a given $H > H^0$ (or similarly, firms' best responses to other firms' prices) may include infinite prices for some final products. This case is beyond our partially aggregative games approach that builds on the generalized Poincaré-Hopf index theorem in Simsek et al. (2007).

8 Conclusion

In this paper, we generalize the aggregative games approach developed in Nocke and Schutz (2018) to analyze price competition among multiproduct firms providing both complements and substitutes. We build a simple network structure to classify this type of market. In this more general setting, we establish equilibrium existence, characterize the pricing equilibria, and show that the classical demands, MNL and CES, continue to have a unique pricing equilibrium under different conditions. We also investigate the properties of the fitting-in functions and show that firms' η-markups and final products' aggregate markups may not be strictly decreasing functions of the aggregator, which measures the competition level and consumer surplus of a market. In our comparative statics analysis, asymmetric networks, where two firms may be involved in different final products, play a crucial role. An increase in the value of the outside option (or the entry of a new competitor) can increase the prices of some existing products and benefit some incumbent firms in an asymmetric network. Moreover, under the commonly used MNL demand, reductions in marginal costs or quality improvements can reduce overall consumer surplus in two-sided asymmetric networks, in contrast to the classical Cournot model, which shows that a merger of complements tends to reduce prices.

Our model applies to several key industries, including airlines, carbonated beverages, and patents, where double marginalization is considered a major problem that reduces market efficiency (Brueckner (2001), Quint (2014b), Luco and Marshall (2020)). While a vertical merger (or a merger of complements) is often viewed as a tool for resolving this issue, the 2020 US Vertical Merger Guidelines and the 2023 US Merger Guidelines both raised concerns regarding potential price increases when multiproduct firms are involved in a vertical merger. Based on our new approach, we can identify the specific market structures under which a merger of complements can improve or harm consumer surplus. Since a merger of complements can be similarly considered a reduction in marginal cost under an $(\mathbf{S})$ or $(\mathbf{A1})$ network, such a merger still improves consumer surplus under these networks, even though it may raise the prices of some final products. We also identify $(\mathbf{A2})$ networks as the primary source of inefficiency induced by a merger of complements. In an $(\mathbf{A2})$ network, even if a merger of complements reduces only the marginal costs of final products, it can trigger sub-

stantial price increases in other final products, ultimately reducing consumer surplus.

This paper also provides a new perspective for understanding platform competition and entry. Antitrust agencies often hope to encourage the entry of new competitors to reduce the market power of the major platforms, as shown in the latest Digital Markets Act in Europe (Seitz (2022)). Major trading platforms like Alibaba or Amazon typically offer their products by building asymmetric networks. They not only provide services to help other sellers sell on their platforms but also provide competing self-owned products in the same market. Furthermore, many large sellers can also sell their products through other channels without a platform, structuring the market as an $(\mathbf{A2})$ network. In contrast, if sellers are small, they may heavily rely on a platform's service, and the network is more like an $(\mathbf{A1})$ network. Our analysis shows that increasing competition in this industry, where platforms tend to have large market shares within a country, can have subtle effects. Prices may not always go down in an $(\mathbf{A2})$ network. More intense competition can benefit small sellers who heavily rely on a platform's service.

Finally, we note that one significant limitation of this model is the assumption that firms compete in linear prices. In a vertical chain, input transactions often involve nonlinear pricing. Similarly, firms also use nonlinear prices to directly sell their products to consumers (e.g., credit cards, truck rentals, and internet services). Nevertheless, our approach remains applicable to certain nonlinear settings. For example, if firms set two-part tariffs simultaneously and consumers pay the aggregate fixed and variable components to buy a combination of products, the pricing game with two-part tariffs is formally equivalent to our baseline model with MNL demand and linear pricing (consistent with Nocke and Schutz (2018)). However, as noted by Moresi and Schwartz (2017) and Miklós-Thal and Shaffer (2021), vertical chains often involve different pricing strategies: Retailers compete in linear prices while input suppliers use two-part tariffs. Our model cannot accommodate these mixed cases. Finding a tractable method to systematically analyze nonlinear pricing in markets with both complements and substitutes is an exciting direction for future research.

Appendix

A Proof of Theorem 1

We first restate several key properties of the $h_i(P_i)$ function:

Lemma 3 (Nocke and Schutz (2018))**.** *Let h be a $\mathcal{C}^3$, strictly decreasing, and log-convex function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Then:*

(a) $\lim_{P\to\infty} Ph'(P) = \lim_{P\to\infty} h'(P) = 0$.

Assuming also that h satisfies Assumption 1, and we define $\gamma(P) = h'^2(P)/h''(P)$, $\rho(P) = h(P)/\gamma(P)$, and $\eta(P) = Ph''(P)/(-h'(P))$. Then:

(b) There exists a unique scalar $\underline{P}$ such that for every $P > 0$, $\eta(P) > 1$ if and only if $P > \underline{P}$. Moreover, $\eta'(P) \geq 0$ for all $P > \underline{P}$.

(c) $\bar{\mu} \equiv \lim_{P\to\infty} \eta(P) > 1$.

(d) For every $P > \underline{P}$, $\gamma'(P) < 0$.

(e) $\lim_{P\to\infty} \gamma(P) = 0$.

(f) If $\lim_{P\to\infty} h(P) = 0$ and $\bar{\mu} < \infty$, then $\lim_{P\to\infty} \rho(P) = \frac{\bar{\mu}}{\bar{\mu}-1}$.

Suppose that the demand system $((h_i)_{i\in\mathcal{N}}, H^0)$ satisfies Assumption 1. With a given network $(\mathcal{F}, \mathcal{N})$ and a vector of marginal costs for all firms $(c_i^k)_{f_k\in\mathcal{F}, i\in G^k}$, we will first illustrate that the generalized first-order conditions are necessary and sufficient for firms' profit maximization from Lemma 4 to Lemma 9. By redefining the η-markup for each firm in this more general setting, we can follow the similar methods developed in Nocke and Schutz (2018) to show these results.

We fix a firm $f_k \in \mathcal{F}$ and a price vector for all other firms $(s_i^l)_{f_l\in\mathcal{F}\setminus\{f_k\}, i\in G^l}$. Since we only consider finite prices, we assume that $s_i^l \in (0, \infty)$ for any $i \in G^l$ and $f_l \in \mathcal{F} \setminus \{f_k\}$. We evaluate the following maximization problem:

$$\max_{s^k\in(0,\infty]^{|G^k|}} \hat{\Pi}^k(s^k, s^{k'}, H^{0'}), \tag{9}$$

where we rewrite firm f_k's profit, $\Pi^k(s^k, s^{k'}, H(s))$, to be

$$\begin{aligned}\hat{\Pi}^k(s^k, s^{k'}, H^{0'}) &= \sum_{i\in G^k} (s_i^k - c_i^k) \frac{-h_i'(s_i^k + \sum_{f_l\in M_i\setminus\{f_k\}} s_i^l)}{H(s)} \\ &= \sum_{i\in G^k} (s_i^k - c_i^k) \frac{-h_i'(s_i^k + \sum_{f_l\in M_i\setminus\{f_k\}} s_i^l)}{\sum_{j\in G^k} h_j(s_j^k + \sum_{f_l\in M_j\setminus\{f_k\}} s_j^l) + H^{0'}},\end{aligned}$$

and $H^{0'} \equiv \sum_{j\in\mathcal{N}\setminus G^k} h_j(P_j) + H^0 > 0$.

Compared to the profit defined in (3), this maximization problem allows firm f_k to set infinite prices in response to other firms' finite prices. Later in Lemma 11, we will show how the optimal solution to this maximization problem can be used to pin down all the finite equilibrium prices. Assumptions 2a and 2b will also be specified after Lemma 11 to ensure that all firms will choose finite prices.

Maximization problem (9) has the following property:

Lemma 4. *Maximization problem (9) has a solution. Moreover, if $(s_i^k)_{i\in G^k}$ solves (9), then $s_i^k \geq c_i^k$ for any $i \in G^k$, and $s_i^k < \infty$ for some $i \in G^k$.*

Proof. The proof is analogous to the proof of Lemma C in Nocke and Schutz (2018). □

We use $(s_j^k, (s_i^k)_{i\in G^k\setminus\{j\}})$ to denote firm f_k's price vector with s_j^k for final product $j \in G^k$ and $(s_i^k)_{i\in G^k\setminus\{j\}}$ for other final products in G^k. The generalized first-order conditions of the maximization problem (9), defined in Nocke and Schutz (2018), is the following:

Definition 2. *We say that the generalized first-order conditions of maximization problem (9) hold at price vector $(\tilde{s}_i^k)_{i\in G^k} \in (0,\infty]^{|G^k|}$ if, for every $j \in G^k$,*

(a) $\frac{\partial \hat{\Pi}^k}{\partial s_j^k}((\tilde{s}_i^k)_{i\in G^k}, s^{k'}, H^{0'}) = 0$ whenever $\tilde{s}_j^k < \infty$, and

(b) $\hat{\Pi}^k((\tilde{s}_i^k)_{i\in G^k}, s^{k'}, H^{0'}) \geq \hat{\Pi}^k((s_j^k, (\tilde{s}_i^k)_{i\in G^k\setminus\{j\}}), s^{k'}, H^{0'})$ for every $s_j^k \in \mathbb{R}_{++}$ whenever $\tilde{s}_j^k = \infty$.

The generalized first-order conditions are clearly necessary for profit maximization:

Lemma 5. *If $(s_i^k)_{i\in G^k} \in (0,\infty]^{|G^k|}$ solves profit maximization problem (9), then the generalized first-order conditions are satisfied at price vector $(s_i^k)_{i\in G^k}$.*

Next, we show that the generalized first-order conditions are also sufficient for global optimality. To see this, we first illustrate that if the first-order conditions for firm f_k are satisfied at a price vector s^k, then this price vector satisfies the common η-markup property discussed in Section 4. We evaluate function $\nu_i^k : s_i^k \to \frac{s_i^k - c_i^k}{P_i}\eta_i(P_i)$ for $i \in G^k$, where $P_i = \sum_{f_l\in M_i} s_i^l$. Let $s_i^{k_{mc}}$ denote the unique solution of equation $\nu_i^k(s_i^k) = 1$. Function $\nu_i^k(s_i^k)$ for $i \in G^k$ differs from function $\nu_i(P_i)$ defined in the discussion preceding Lemma E in Nocke and Schutz (2018) because it includes additional terms for prices of firm f_k's complementary products in the denominator. However, since these terms are positive constants that only enter through P_i for $i \in G^k$, the function $\nu_i^k(s_i^k)$ retains the two properties of $\nu_i(P_i)$ in Nocke and Schutz (2018): By Lemma 3, $\nu_i^k(s_i^k)$ is strictly increasing when $\nu_i^k(s_i^k) > 1$, and $\lim_{s_i^k\to\infty} \nu_i^k(s_i^k) = \bar{\mu}_i > 1$. Also, since our setting includes complementary firms, $\nu_i^k(s_i^k)$ is specific to firm $f_k \in M_i$. We have the following lemma:

Lemma 6. *For any $i \in G^k$ and $\mu^k \in (1, \bar{\mu}_i)$, the equation $\nu_i^k(s_i^k) = \mu^k$ has a unique solution on the interval $(0,\infty)$, denoted by $r_i^k(\mu^k)$. The function $r_i^k(\mu^k)$ is strictly increasing and $\mathcal{C}^1$ on $(1,\bar{\mu}_i)$, which satisfies $\lim_{\mu^k\to 1} r_i^k(\mu^k) = s_i^{k_{mc}}$, $\lim_{\mu^k\to\bar{\mu}_i} r_i^k(\mu^k) = \infty$, and*

$$r_i^{k\prime}(\mu^k) = \frac{\gamma_i(r_i^k(\mu^k) + \sum_{f_l\in M_i\setminus\{f_k\}} s_i^l)}{(\mu^k - 1)h_i'(r_i^k(\mu^k) + \sum_{f_l\in M_i\setminus\{f_k\}} s_i^l) - \mu^k\gamma_i'(r_i^k(\mu^k) + \sum_{f_l\in M_i\setminus\{f_k\}} s_i^l)} > 0. \tag{10}$$

Proof. The proof is analogous to the proof of Lemma E in Nocke and Schutz (2018). □

We extend the function r_i^k by continuity: $r_i^k(1) = s_i^{k_{mc}}$, and $r_i^k(\mu^k) = \infty$ for any $\mu^k \geq \bar{\mu}_i$. Now we formalize the definition of the common η-markup property.

Definition 3. *The price vector* $(s_i^k)_{i\in G^k} \in (0,\infty]^{|G^k|}$ *satisfies the common* η*-markup property if there exists a scalar* $\mu^k \geq 1$ *such that* $s_i^k = r_i^k(\mu^k)$ *for any* $i \in G^k$.

We define $\bar{\mu}^k \equiv \max_{i\in G^k} \bar{\mu}_i$ and extend γ_i by continuity at infinity such that $\gamma_i(\infty) = 0$ for any $i \in \mathcal{N}$ (by Lemma 3). We can further simplify the generalized first-order conditions by using the common η-markup property:

Lemma 7. *Suppose that the generalized first-order conditions for optimization problem (9) hold at price vector* $(s_i^k)_{i\in G^k} \in (0,\infty]^{|G^k|}$. *Then* $(s_i^k)_{i\in G^k}$ *satisfies the common* η*-markup property. The common* η*-markup,* μ^k*, solves the following equation on* $(1,\bar{\mu}^k)$*:*

$$\mu^k = 1 + \mu^k \frac{\sum_{j\in G^k} \gamma_j(r_j^k(\mu^k) + \sum_{f_l\in M_j\setminus\{f_k\}} s_j^l)}{\sum_{j\in G^k} h_j(r_j^k(\mu^k) + \sum_{f_l\in M_j\setminus\{f_k\}} s_j^l) + H^{0'}}. \tag{11}$$

Also, $\hat{\Pi}^k(s^k, s^{k'}, H^{0'}) = \mu^k - 1$.

Proof. The proof is analogous to the proof of Lemma F in Nocke and Schutz (2018). □

Next, we show that equation (11) has a unique solution in μ^k for firm f_k.

Lemma 8. *There exists a unique* $\mu^k \in (1,\bar{\mu}^k)$ *that solves equation* (11).

Proof. The proof is analogous to the proof of Lemma G in Nocke and Schutz (2018). □

With all these results, we can conclude that the generalized first-order conditions are necessary and sufficient for global optimality in solving maximization problem (9).

Lemma 9. *Profit maximization problem (9) has a unique solution. The generalized first-order conditions of this problem are both necessary and sufficient for global optimality. The optimal price vector satisfies the common* η*-markup property, and the corresponding* η*-markup,* μ^{k*}*, is the unique solution of equation (11). The optimal profit is* $\mu^{k*} - 1$.

Proof. Based on the results from Lemma 4 to Lemma 8, the proof can be constructed following the argument used in the proof of Lemma H in Nocke and Schutz (2018). □

We now focus on proving the existence of a pricing equilibrium. Based on Lemma 9, we know that each firm $f_k \in \mathcal{F}$ chooses the optimal η-markup in response to other firms' prices. For each final product $i \in \mathcal{N}$, we add all η-markups of firms providing intermediates for final product i together to derive the aggregate markup, Ψ_i, where

$$\sum_{f_k\in M_i} v_i^k(s_i^k) = \sum_{f_k\in M_i} \mu^k = \frac{P_i - C_i}{P_i}\eta_i(P_i) = \Psi_i. \tag{12}$$

Similar to Lemma 6, we can define a function $w_i : P_i \to \frac{P_i - C_i}{P_i}\eta_i(P_i)$, and P_i^{mc} is the unique solution for $w_i(P_i) = 1$. The following lemma shows some important properties about function $w_i(P_i)$.

Lemma 10. *For any $i \in \mathcal{N}$ and $\Psi_i \in (1, \bar{\mu}_i)$, the equation $w_i(P_i) = \Psi_i$ has a unique solution on the interval $(0, \infty)$, denoted $t_i(\Psi_i)$. Moreover, $t_i(\cdot)$ is strictly increasing and $\mathcal{C}^1$ on $(1, \bar{\mu}_i)$ and has the properties that $\lim_{\Psi_i \to 1} t_i(\Psi_i) = P_i^{mc}$, $\lim_{\Psi_i \to \bar{\mu}_i} t_i(\Psi_i) = \infty$, and*

$$t_i'(\Psi_i) = \frac{\gamma_i(t_i(\Psi_i))}{(\Psi_i - 1)h_i'(t_i(\Psi_i)) - \Psi_i \gamma_i'(t_i(\Psi_i))} > 0.$$

Proof. The proof is analogous to the proof of Lemma E in Nocke and Schutz (2018). □

Similarly, we extend this function t_i for any $i \in \mathcal{N}$ by continuity: $t_i(\Psi_i) = P_i^{mc}$ for any $0 \leq \Psi_i \leq 1$, and $t_i(\Psi_i) = \infty$ for any $\Psi_i \geq \bar{\mu}_i$.

Based on Lemmas 9 and 10, and the assumption that firms' prices are finite, we summarize the problem of the existence of a pricing equilibrium in the following way.

Lemma 11. *There exists a pricing equilibrium if and only if there exists a profile of η-markups $(\mu^k)_{f_k \in \mathcal{F}}$, which satisfies $1 < \mu^k < \bar{\mu}^k$ for every $f_k \in \mathcal{F}$, $|M_i| < \Psi_i < \bar{\mu}_i$ for every $i \in \mathcal{N}$, and*

$$\mu^k = 1 + \mu^k \frac{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i))}{\sum_{j \in \mathcal{N}} h_j(t_j(\Psi_j)) + H^0}, \quad \forall f_k \in \mathcal{F}. \tag{13}$$

Proof. Let $s^* = (s_i^{k*})_{f_k \in \mathcal{F}, i \in G^k}$ be a pricing equilibrium. Then, the generalized first-order conditions and the common η-markup property are satisfied for every firm $f_k \in \mathcal{F}$ with s^*. By Lemma 9, s^* corresponds to a profile of η-markups $(\mu^{k*})_{k \in \mathcal{F}}$, which has the property that for any $f_k \in \mathcal{F}$, $1 < \mu^{k*} < \bar{\mu}^k$, and

$$\mu^{k*} = 1 + \mu^{k*} \frac{\sum_{i \in G^k} \gamma_i(\sum_{f_l \in M_i} r_i^l(\mu^{l*}))}{\sum_{j \in \mathcal{N}} h_j(\sum_{f_l \in M_j} r_j^l(\mu^{l*})) + H^0}.$$

Since $\sum_{f_l \in M_i} r_i^l(\mu^{l*}) = \sum_{f_l \in M_i} s_i^{l*} = P_i^* = t_i(\Psi_i^*)$ for any $i \in \mathcal{N}$, the equation above can be rewritten as

$$\mu^{k*} = 1 + \mu^{k*} \frac{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i^*))}{\sum_{j \in \mathcal{N}} h_j(t_j(\Psi_j^*)) + H^0}, \quad \forall f_k \in \mathcal{F}.$$

Suppose that there exists a profile of η-markups, $(\mu^{k*})_{f_k \in \mathcal{F}}$, such that $1 < \mu^{k*} < \bar{\mu}^k$ for any $f_k \in \mathcal{F}$, $|M_i| < \Psi_i^* < \bar{\mu}_i$ for any $i \in \mathcal{N}$, and equation (13) is satisfied. Then, by Lemma 10, we derive the corresponding finite price for each final product i, where $P_i^* = t_i(\Psi_i^*)$. Based on the definition of η-markup, as shown in Lemma 6, we can determine a unique price vector $s^* = (s_i^{k*})_{f_k \in \mathcal{F}, i \in G^k}$. Since $P_i^* = t_i(\Psi_i^*) = \sum_{f_l \in M_i} r_i^l(\mu^{l^*})$ for any $i \in \mathcal{N}$, the generalized first-order conditions are satisfied for every firm $f_k \in \mathcal{F}$, according to Lemma 9, and s^* is a pricing equilibrium. □

This equilibrium existence problem is equivalent to finding an aggregator $H > H^0$ and a profile of η-markups $(\mu^k)_{f_k \in \mathcal{F}}$, such that $1 < \mu^k < \bar{\mu}^k$ and $|M_i| < \Psi_i < \bar{\mu}_i$ for

any $i \in \mathcal{N}$ and $f_k \in \mathcal{F}$, $H = \sum_{j\in\mathcal{N}} h_j(t_j(\Psi_j)) + H^0$, and

$$\mu^k = 1 + \mu^k \frac{\sum_{i\in G^k} \gamma_i(t_i(\Psi_i))}{H} \equiv \xi^k(\mu^k, \mu^{k'}, H), \quad \forall f_k \in \mathcal{F}. \tag{14}$$

In equation (14), $\mu^{k'} = (\mu^l)_{f_l \in f_{k'}}$ is a vector of η-markups for other firms providing complements for firm f_k. As discussed in Section 4, the aggregative games approach developed in Nocke and Schutz (2018) cannot be directly applied to show the existence of a pricing equilibrium. However, in this more general setting, when an aggregator $H \geq H^0$ is given, firms' η-markups and final products' aggregate markups from different separable components are independent of each other. We first show that the system of equations (14) for firms within the same separable component uniquely determines these firms' η-markups and the corresponding aggregate markups with a given aggregator $H \geq H^0$.

To obtain this result, we now assume that the demand system $((h_i)_{i\in\mathcal{N}}, H^0)$ also satisfies Assumption 2a or Assumption 2b. For Assumption 2a, we define $\underline{H}^0$ here. Consider the following equation for firm $f_k \in \mathcal{F}$:

$$\phi^k(\mu^k, H) \equiv 1 - \left(\frac{\sum_{j\in G^k} \gamma_j(t_j(\mu^k + |M_j| - 1))}{H} + \frac{1}{\mu^k}\right) = 0. \tag{15}$$

By Lemmas 3 and 10, $\phi^k(\mu^k, H)$ is continuous on $(1,\infty) \times \mathbb{R}_{++}$, and $\phi^k(\mu^k, H)$ is strictly increasing in μ^k for any $H > 0$. Since $|M_i| < \bar{\mu}_i$ for any $i \in \mathcal{N}$ under Assumption 2a and by Lemmas 3 and 10, $\lim_{\mu^k \to 1} \phi^k(\mu^k, H) < 0$ and $\lim_{\mu^k \to \bar{\mu}^k} \phi^k(\mu^k, H) > 0$. Therefore, there exists a unique solution of equation (15) in μ^k on $(1, \bar{\mu}^k)$ for any $H > 0$, denoted $\delta^k(H)$. Similarly, equation (15) has a unique solution in H on $(0,\infty)$ for any $\mu^k \in (1, \max_{i\in G^k}(\bar{\mu}_i + 1 - |M_i|))$. Let $\underline{\mu}^k = \min(\{\frac{\bar{\mu}_i}{|M_i|}\}_{i\in G^k})$ and $G^{\hat{k}} = \{j \in G^k : \bar{\mu}_j < \infty\}$ for any $f_k \in \mathcal{F}$. Clearly, $\underline{\mu}^k \in (1, \bar{\mu}^k]$, and $\underline{\mu}^k = \bar{\mu}^k$ if and only if either $G^{\hat{k}} = \varnothing$ or $\bar{\mu}_i = \bar{\mu}_j < \infty$ for any $i, j \in G^k$ and $f_{k'} = \varnothing$ (which means that $|M_i| = 1$ for every $i \in G^k$). If $\underline{\mu}^k \in (1, \bar{\mu}^k)$ for a firm $f_k \in \mathcal{F}$, then, with $\mu^k = \underline{\mu}^k$, there is a unique solution of equation (15) in H on $(0,\infty)$, denoted $\tilde{H}^k$. If $\underline{\mu}^k = \bar{\mu}^k$ for a firm $f_k \in \mathcal{F}$, then we set $\tilde{H}^k = 0$. In Assumption 2a, $\underline{H}^0$ is chosen to be the maximum of $(\tilde{H}^k)_{f_k\in\mathcal{F}}$.[34] If $\bar{\mu}_i = \infty$ for every $i \in \mathcal{N}$, then $\underline{H}^0 = 0$, which is equivalent to Assumption 2b. Finally, since $\phi^k(\mu^k, H)$ is strictly increasing in μ^k and H on $(1,\infty) \times \mathbb{R}_{++}$, $\delta^k(H)$ is strictly decreasing in H. Hence, for any $H > \underline{H}^0$ and $f_k \in \mathcal{F}$, $\delta^k(H) < \frac{\bar{\mu}_i}{|M_i|}$ for every $i \in G^k$.

We now evaluate the solution of equations (14) in $(\mu^k)_{f_k\in\mathcal{F}}$ with $H \geq H^0 > \underline{H}^0$.

[34] Alternatively, we can set $\underline{H}^0 = \max_{i\in\mathcal{N}} \tilde{H}_i$, where $\tilde{H}_i$ is the solution of $\sum_{f_k\in M_i} \delta^k(\tilde{H}_i) = \bar{\mu}_i$ for any $i \in \mathcal{N}$ when $\sum_{f_k\in M_i} \bar{\mu}^k > \bar{\mu}_i$ and $\tilde{H}_i = 0$ when $\sum_{f_k\in M_i} \bar{\mu}^k \leq \bar{\mu}_i$. The proof of Lemma 12 still holds under this version of $\underline{H}^0$ by replacing the box $\prod_{f_k\in L_1}[1, b_k]$ with $\prod_{f_k\in L_1}[1, \delta^k(H) + \epsilon]$ for a sufficiently small $\epsilon > 0$. Only the proof of Proposition 9 (c) relies on the $\underline{H}^0$ defined based on the firm-level bounds.

Lemma 12. *For any $S_\alpha \in S$ and $H > \underline{H}^0$, a system of equations (14) for all firms in L_α has a unique solution in $(\mu^k)_{f_k \in L_\alpha}$ on $\prod_{f_k \in L_\alpha}(1, \bar{\mu}^k)$. We denote the solution by $(g^k(H))_{f_k \in L_\alpha}$, where $g^k(H)$ is differentiable on $(\underline{H}^0, \infty)$ for any $f_k \in L_\alpha$. Also, for any $i \in S_\alpha$, we denote $\sum_{f_l \in M_i} g^l(H)$ by $m_i(H)$, where $|M_i| < m_i(H) < \bar{\mu}_i$.*

Proof. Given a network of $(\mathcal{F}, \mathcal{N})$ with a total of n final products, suppose that the network can be partitioned into $y \geq 1$ separable components, denoted $S = \{S_1, S_2, ..., S_y\}$. Clearly, changes in the system of equations (14) in one separable component do not affect aggregate markups in other separable components when an aggregator $H > \underline{H}^0$ is given. Without loss of generality, we study equations (14) for firms providing intermediates for S_1, where $S_1 = \{1, 2, ..., n_1\}$ with $n_1 \geq 1$. Let $L_1 = \{f_1, f_2, ..., f_q\} \subseteq \mathcal{F}$ with $q \geq 1$ denote the set of firms in S_1.

We first show that the right-hand side of equation (14) for any firm in L_1 with $(\mu^k)_{f_k \in L_1} \in [1, \infty)^{|L_1|}$ is bounded. Let $\mathbf{1}^{k'} = (1)_{|f_{k'}|}$ denote a vector of ones with $|f_{k'}| \geq 0$ entries for firm $f_k \in L_1$. By Lemmas 3 and 10, we know that, for any $(\mu^l)_{f_l \in L_1} \in [1, \infty)^{|L_1|}$ and $f_k \in L_1$,

$$\xi^k(\mu^k, \mathbf{1}^{k'}, H) = 1 + \mu^k \frac{\sum_{i \in G^k} \gamma_i(t_i(\mu^k + |M_i| - 1))}{H} \geq \xi^k(\mu^k, \mu^{k'}, H),$$

where $\xi^k(\mu^k, \mu^{k'}, H)$ was defined in equation (14). Since, by Lemmas 3 and 10 and the extension of t_i for any $i \in \mathcal{N}$ by continuity, $\xi^k(\mu^k, \mathbf{1}^{k'}, H)$ is a continuous function in μ^k on $[1, \infty)$, and, for any $i \in G^k$,

$$\begin{aligned}
\mu^k \gamma_i(t_i(\mu^k + |M_i| - 1)) &\leq (\mu^k + |M_i| - 1) \frac{t_i(\mu^k + |M_i| - 1)(-h_i'(t_i(\mu^k + |M_i| - 1)))}{\eta_i(t_i(\mu^k + |M_i| - 1))} \\
&= (t_i(\mu^k + |M_i| - 1) - C_i)(-h_i'(t_i(\mu^k + |M_i| - 1))) \\
&\leq t_i(\mu^k + |M_i| - 1)(-h_i'(t_i(\mu^k + |M_i| - 1)))
\end{aligned}$$

where $\lim_{\mu^k \to \infty} t_i(\mu^k + |M_i| - 1)(-h_i'(t_i(\mu^k + |M_i| - 1))) = 0$.

Then, $\xi^k(\mu^k, \mathbf{1}^{k'}, H)$ has a finite limit at infinity, which implies that this continuous function $\xi^k(\mu^k, \mathbf{1}^{k'}, H)$ on $[1, \infty)$ is bounded, with supremum $\bar{\xi}^k(H)$, for any $H > \underline{H}^0$. This then implies that, for every $f_k \in L_1$, $1 \leq \xi^k(\mu^k, \mu^{k'}, H) \leq \xi^k(\mu^k, \mathbf{1}^{k'}, H) \leq \bar{\xi}^k(H)$, where $H > \underline{H}^0$ and $(\mu^k)_{f_k \in L_1} \in [1, \infty)^{|L_1|}$. Therefore, with a given $H > \underline{H}^0$, $(\xi^k(\mu^k, \mu^{k'}, H))_{f_k \in L_1}$ is a function in $(\mu^k)_{f_k \in L_1}$ with fixed points only in $\prod_{f_k \in L_1}[1, \bar{\xi}^k(H)]$.

Next, we rule out points that cannot be solutions to the system of equations (14). Under Assumption 2a, our definition of $\underline{H}^0$ implies that the solution of equations (14) with $H > \underline{H}^0$ for firms in L_1 must satisfy $\mu^k < \underline{\mu}^k$ for any $f_k \in L_1$ and $\Psi_i < \bar{\mu}_i$ for any $i \in S_1$. Then, in any solution of equations (14), $\gamma_i(t_i(\Psi_i)) > 0$ for any $i \in S_1$, which implies that $\mu^k > 1$ for any $f_k \in L_1$. Since we have shown that $\bar{\xi}^k(H) \geq \xi^k(\mu^k, \mu^{k'}, H)$ for every $f_k \in L_1$, $H > \underline{H}^0$, and $(\mu^k)_{f_k \in L_1} \in [1, \infty)^{|L_1|}$, the solution must

be in $\prod_{f_k\in L_1}(1,\min(\underline{\mu}^k,1+\bar{\xi}^k(H)))\subseteq\prod_{f_k\in L_1}(1,\bar{\mu}^k)$. Under Assumption 2b, since $\underline{\mu}^k=\infty$ for any $f_k\in L_1$, the solution must be in $\prod_{f_k\in L_1}(1,1+\bar{\xi}^k(H))$ for any $H>\underline{H}^0$.

We now show the uniqueness of the solution of equations (14) for firms in L_1 in $(\mu^k)_{f_k\in L_1}\in\prod_{f_k\in L_1}[1,b_k]\equiv\mathbf{J}^{L_1}$, where $b_k\equiv\min(\underline{\mu}^k,1+\bar{\xi}^k(H))$, under Assumption 2a. This analysis also holds under Assumption 2b. We rewrite equations (14) with a given $H>\underline{H}^0$ for firms in L_1 and define a function $\tau((\mu^k)_{f_k\in L_1},H)$ from $\mathbf{J}^{L_1}\times(\underline{H}^0,\infty)$ to $\mathbb{R}^{|L_1|}$:

$$\tau((\mu^k)_{f_k\in L_1},H)=(1-\frac{1}{\mu^k}-\frac{\sum_{j\in G^k}\gamma_j(t_j(\Psi_j))}{H})_{f_k\in L_1}$$

$(\mu^k)_{f_k\in L_1}$ solves equations (14) for firms in L_1 if and only if $\tau((\mu^k)_{f_k\in L_1},H)=0$. We will illustrate that $\tau((\mu^k)_{f_k\in L_1},H)=0$ has a unique solution in $(\mu^k)_{f_k\in L_1}$ on the compact set $\mathbf{J}^{L_1}$ by applying the generalized Poincaré-Hopf index theorem (Simsek et al. (2007)). Note that $\mathbf{J}^{L_1}$ is a nonempty compact convex region defined by the constraints $1-\mu^k\leq 0$ and $\mu^k-b_k\leq 0$ for every $f_k\in L_1$, which satisfy the LICQ in Assumption 2.1 of Simsek et al. (2007), so we can apply their Theorem 3.1 with $\chi(\mathbf{J}^{L_1})=1$.

First, $\tau((\mu^k)_{f_k\in L_1},H)$ is continuous on $\prod_{f_k\in L_1}(0,b_k+1)$, which is an open set containing $\mathbf{J}^{L_1}$, by Lemmas 3 and 10 and the extension of function t_i for any $i\in\mathcal{N}$. Second, $\tau((\mu^k)_{f_k\in L_1},H)$ has no generalized critical point on the boundary of $\mathbf{J}^{L_1}$. By Definition 3.1 of Simsek et al. (2007), such a point $x\in\mathbf{J}^{L_1}$ satisfies $-\tau(x,H)\in N_{\mathbf{J}^{L_1}}(x)$, which requires that $\tau^k(x,H)\geq 0$ whenever $\mu^k=1$ and $\tau^k(x,H)\leq 0$ whenever $\mu^k=b_k$, where $\tau^k(x,H)$ denotes the kth component of function $\tau((\mu^k)_{f_k\in L_1},H)$. Neither can hold. If $\mu^k=1$, then for every $i\in G^k$, $\Psi_i\leq 1+(|M_i|-1)\frac{\bar{\mu}_i}{|M_i|}<\bar{\mu}_i$ under Assumption 2a, which implies that $\gamma_i(t_i(\Psi_i))>0$ and $\tau^k((\mu^k)_{f_k\in L_1},H)=1-\frac{1}{\mu^k}-\frac{\sum_{j\in G^k}\gamma_j(t_j(\Psi_j))}{H}<0$. If $\mu^k=b_k$, then, since $\mu^l\geq 1$ for any $f_l\in L_1$ and $\gamma_i(t_i(\Psi_i))$ is nonincreasing for any $i\in G^k$ on $(1,\infty)$, $\tau^k((\mu^k)_{f_k\in L_1},H)\geq\phi^k(b_k,H)$ (defined in equation (15)). When $b_k=\underline{\mu}^k$, $\phi^k(b_k,H)>0$ for any $H>\underline{H}^0$ under Assumption 2a. When $b_k=1+\bar{\xi}^k(H)$, we have $\tau^k((\mu^k)_{f_k\in L_1},H)=1-\frac{\xi^k(\mu^k,\mu^{k'},H)}{\mu^k}\geq\frac{1}{\mu^k}>0$ because $\xi^k(\mu^k,\mu^{k'},H)\leq\bar{\xi}^k(H)$. Therefore, the generalized critical points of $\tau((\mu^k)_{f_k\in L_1},H)$ over $\mathbf{J}^{L_1}$ with a given $H>\underline{H}^0$ are in $\prod_{f_k\in L_1}(1,b_k)$, where $\Psi_i\in(1,\bar{\mu}_i)$ for any $i\in S_1$, which also ensures that $\tau((\mu^k)_{f_k\in L_1},H)$ is continuously differentiable by Lemmas 3 and 10. Since $I(x)=\varnothing$ at interior points, each critical point is a zero of $\tau((\mu^k)_{f_k\in L_1},H)$, is complementary, and has index sign $\det(\nabla\tau((\mu^k)_{f_k\in L_1},H))$ (by Definitions 3.1-3.2 of Simsek et al. (2007)). Finally, all we need to show is that $\det(\nabla\tau((\mu^k)_{f_k\in L_1},H))>0$ at every zero of function $\tau((\mu^k)_{f_k\in L_1},H)$, which means that every critical point is nondegenerate with index one. Then Theorem 3.1 shows that $\sum_{x\in\mathrm{Cr}(\tau,\mathbf{J}^{L_1})}\mathrm{ind}_\tau(x)=\chi(\mathbf{J}^{L_1})=1$, and $\tau((\mu^k)_{f_k\in L_1},H)=0$ has a unique solution in $\mathbf{J}^{L_1}$.

We now prove $\det(\nabla\tau((\mu^k)_{f_k\in L_1},H))>0$ at every zero of function $\tau((\mu^k)_{f_k\in L_1},H)$.

For any $(\mu^k)_{f_k\in L_1} \in \prod_{f_k\in L_1}(1, \min(\underline{\mu}^k, 1+\bar{\xi}^k(H)))$,

$$\nabla\tau((\mu^k)_{f_k\in L_1}, H) = \begin{bmatrix} \frac{-\sum_{j\in G^1}\gamma_j' t_j'}{H} + (\frac{1}{\mu^1})^2 & \frac{-\sum_{j\in(G^1\cap G^2)}\gamma_j' t_j'}{H} & \cdots & \frac{-\sum_{j\in(G^1\cap G^q)}\gamma_j' t_j'}{H} \\ \frac{-\sum_{j\in(G^2\cap G^1)}\gamma_j' t_j'}{H} & \frac{-\sum_{j\in G^2}\gamma_j' t_j'}{H} + (\frac{1}{\mu^2})^2 & \cdots & \frac{-\sum_{j\in(G^2\cap G^q)}\gamma_j' t_j'}{H} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\sum_{j\in(G^q\cap G^1)}\gamma_j' t_j'}{H} & \frac{-\sum_{j\in(G^q\cap G^2)}\gamma_j' t_j'}{H} & \cdots & \frac{-\sum_{j\in G^q}\gamma_j' t_j'}{H} + (\frac{1}{\mu^q})^2 \end{bmatrix}.$$

This matrix $\nabla\tau((\mu^k)_{f_k\in L_1}, H)$ is symmetric. For final product $i \in S_1$, let x_i be a vector with q entries, where the kth row of x_i is equal to 1 if firm f_k provides intermediates for final product i, and 0 otherwise. Then, we define matrix E_i for final product $i \in S_1$, to be

$$E_i = \frac{-\gamma_i' t_i'}{H}\begin{bmatrix} \chi_{G^1}(i) & \chi_{G^1}(i)\chi_{G^2}(i) & \cdots & \chi_{G^1}(i)\chi_{G^q}(i) \\ \chi_{G^2}(i)\chi_{G^1}(i) & \chi_{G^2}(i) & \cdots & \chi_{G^2}(i)\chi_{G^q}(i) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{G^q}(i)\chi_{G^1}(i) & \chi_{G^q}(i)\chi_{G^2}(i) & \cdots & \chi_{G^q}(i) \end{bmatrix} = \frac{-\gamma_i' t_i'}{H}(x_i x_i^T),$$

where $\chi_{G^k}(i)$ is an indicator function, which is equal to 1 when final product $i \in G^k$. E_i identifies the firms that are involved in the production of final product i. Note that for any vector $y \in \mathbb{R}^q$, $y^T E_i y = \frac{-\gamma_i' t_i'}{H}(y^T x_i)^2 \geq 0$, which implies that matrix E_i is positive-semidefinite. $\nabla\tau((\mu^k)_{f_k\in L_1}, H)$ can be rewritten as

$$\nabla\tau((\mu^k)_{f_k\in L_1}, H) = \text{Diag}((\frac{1}{\mu^1})^2, (\frac{1}{\mu^2})^2, \cdots, (\frac{1}{\mu^q})^2) + \sum_{i\in S_1} E_i,$$

where $\text{Diag}((\frac{1}{\mu^1})^2, (\frac{1}{\mu^2})^2, \cdots, (\frac{1}{\mu^q})^2)$ is a q by q diagonal matrix with $(\frac{1}{\mu^k})^2$ on its kk-entry. Since $\nabla\tau((\mu^k)_{f_k\in L_1}, H)$ is a summation of positive semi-definite matrices $(E_i)_{i\in S_1}$ and a positive definite matrix $\text{Diag}((\frac{1}{\mu^1})^2, (\frac{1}{\mu^2})^2, \cdots, (\frac{1}{\mu^q})^2)$, $\det(\nabla\tau((\mu^k)_{f_k\in L_1}, H)) > 0$ for any $(\mu^k)_{f_k\in L_1} \in \mathbf{J}^{L_1}$ that satisfies $\tau((\mu^k)_{f_k\in L_1}, H) = 0$. Therefore, by Theorem 3.1 of Simsek et al. (2007), for every $H > \underline{H}^0$, the system of equations (14) for firms in L_1 has a unique solution in $(\mu^k)_{f_k\in L_1}$, denoted $(g^k(H))_{f_k\in L_1}$. The finding that for any solution $(\mu^k)_{f_k\in L_1} \in \mathbf{J}^{L_1}$, $\det(\nabla\tau((\mu^k)_{f_k\in L_1}, H)) > 0$, also enables us to directly apply the implicit function theorem to conclude that $g^k(H)$ is differentiable on $(\underline{H}^0, \infty)$ for any $f_k \in L_1$.

Based on the definition of an aggregate markup shown in (12), the aggregate markup for any final product $i \in S_1$, denoted $m_i(H) = \sum_{f_l\in M_i} g^l(H)$, is a differentiable function of the aggregator $H > \underline{H}^0$. Since $(g^k(H))_{f_k\in L_1} \in \prod_{f_k\in L_1}(1, \min(\underline{\mu}^k, 1+\bar{\xi}^k(H)))$, $|M_i| < m_i(H) < \bar{\mu}_i$ for every $i \in S_1$. □

By Lemma 12, to show the existence of a pricing equilibrium, we just need to prove that there exists an $H^* > H^0 > \underline{H}^0$ such that:

$$\Gamma(H^*) = H^0 + \sum_{j \in \mathcal{N}} h_j(t_j(m_j(H^*))) = H^0 + \sum_{S_\alpha \in S} \sum_{j \in S_\alpha} h_j(t_j(m_j(H^*))) = H^*$$

Lemma 13. *There exists $H^* \in (H^0, \infty)$ such that $\Gamma(H^*) = H^*$.*

Proof. By Lemmas 3, 10, and 12, $\Gamma(H)$ is a continuous function on $[H^0, \infty)$ for every $H^0 > \underline{H}^0 \geq 0$. With any $H \geq H^0$, $1 < m_i(H) < \bar{\mu}_i$ for every $i \in \mathcal{N}$ by Lemma 12. This means that $0 < h_i(t_i(m_i(H))) < h_i(C_i)$ by Lemma 10. Then, for every $H \geq H^0$, $\Gamma(H) > H^0$ and $\Gamma(H) < H^0 + \sum_{i \in \mathcal{N}} h_i(C_i)$. Therefore, $\Gamma(H)$ is a continuous function from $[H^0, H^0 + \sum_{i \in \mathcal{N}} h_i(C_i)]$ to itself. By Brouwer's fixed-point theorem and the fact that $\Gamma(H^0) > H^0$, we obtain the existence of H^* in (H^0, ∞). □

This series of lemmas completes the proof of Theorem 1.

A.1 Proof of Proposition 1

Proof. The proof of Theorem 1 illustrated that the set of equilibrium aggregators is the set of solutions of the equation $\Gamma(H) = H$ in H on (H^0, ∞). If H^* is an equilibrium aggregator, then, by Lemmas 9, 10, and 12, the equilibrium price of a final product $i \in \mathcal{N}$ is $t_i(m_i(H^*))$, and the equilibrium profit of firm $f_k \in \mathcal{F}$ is $g^k(H^*) - 1$. The equilibrium consumer surplus is $\log(H^*)$ under the demand system specified in equation (1). □

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Competition with Complements and Substitutes

—Online Appendix—

B Proof of Proposition 2

Proof. Note that Lemma 12 shows that firm f_k's fitting-in function g^k is differentiable on (H^0, ∞). All we need to do is show that, when S_α is $(\mathbf{S})$ and $f_k \in L_\alpha$, $g^k(H)$ is strictly decreasing on (H^0, ∞) and $\lim_{H\to\infty} g^k(H) = 1$.

We first argue that if S_α is $(\mathbf{S})$, then $g^k(H) = g^l(H)$ for any $f_k, f_l \in L_\alpha$. By the definition of an $(\mathbf{S})$ network and a separable component, if $f_k, f_l \in L_\alpha$, then $G^k = G^l = S_\alpha$. By Lemma 12, we see that $g^k(H) = g^l(H)$ for any $f_k, f_l \in L_\alpha$, where

$$g^k(H) = \frac{H}{H - \sum_{j\in G^k} \gamma_j(t_j(m_j(H)))}.$$

Next, with a given $H > H^0$, by Lemma 12, the aggregate markup $\Psi_i = m_i(H)$ for any $i \in S_\alpha$ satisfies the following equation:

$$\Psi_i = \sum_{f_l\in M_i} g^l(H) = \sum_{f_l\in M_i} \frac{H}{H - \sum_{j\in G^l} \gamma_j(t_j(\Psi_j))}. \tag{16}$$

If final products $i, j \in G^k$ and $f_k \in L_\alpha$, then, since $S_\alpha = G^k$ is $(\mathbf{S})$, we have $M_i = M_j$, and equation (16) implies that $m_i(H) = |M_i| g^k(H) = m_j(H) \equiv \Psi$, where $1 \le |M_i| < \Psi < \bar{\mu}_i$ for any $i \in S_\alpha$. By rearranging equation (16), we obtain

$$\Psi - \sum_{f_l\in M_i} \frac{H}{H - \sum_{j\in G^l} \gamma_j(t_j(\Psi))} = 0. \tag{17}$$

Since the left-hand side of equation (17) is strictly increasing in Ψ and H by Lemmas 3, 10 and 12, $m_i(H)$ for any $i \in S_\alpha$ is strictly decreasing in H on (H^0, ∞). The property that $g^k(H) = \frac{m_i(H)}{|M_i|}$ for any $f_k \in L_\alpha$ implies that $g^k(H)$ is also strictly decreasing in H on (H^0, ∞). Taking the limit as H tends to infinity in equation (17) implies that $\lim_{H\to\infty} m_i(H) = |M_i|$ for any $i \in S_\alpha$ and $\lim_{H\to\infty} g^k(H) = 1$ for any $f_k \in L_\alpha$. □

B.1 Proof of Lemma 1

Proof. Suppose that firms $f_k, f_l \in \mathcal{F}$ have complements (i.e., $G^k \cap G^l \neq \varnothing$). By rearranging equation (14), firm f_k's η-markup with an $H > H^0$ satisfies

$$\frac{\mu^k - 1}{\mu^k} - \frac{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i))}{H} = 0. \tag{18}$$

By Lemmas 3 and 10, the left-hand side of equation (18) is strictly increasing in μ^k on $(1, \infty)$. Then, equation (18) implicitly defines firm f_k's best response to $\mu^{k'} \in (1, \infty)^{|f_{k'}|}$, which we denote $R^k(\mu^{k'})$. Under Assumption 2a, the possible solution of μ^l for equation (14) with $H > H^0$ is on the interval $(1, \underline{\mu}^l)$ and $\gamma_i > 0$ for any $i \in G^k$. Then, by Lemmas 3 and 10, the left-hand side of equation (18) is also strictly increasing in μ^l on $(1, \underline{\mu}^l)$. Therefore, $R^k(\mu^{k'})$ is strictly decreasing in μ^l on $(1, \underline{\mu}^l)$. □

B.2 Proof of Lemma 2

Proof. When S_α is (**A1**), we first show that there always exists a firm $f_k \in L_\alpha$ providing intermediate products for all final products in S_α. Assume for a contradiction that there does not exist such a firm $f_k \in L_\alpha$ with $G^k = S_\alpha$.[35] For final product $i \in S_\alpha$, since S_α is (**A1**), we pick the largest firm f_l in M_i such that $G^m \subseteq G^l$ for any $f_m \in M_i$. Then, $G^l \neq S_\alpha$ and $G^l \subseteq S_\alpha$. Since S_α is (**A1**) and f_l is the largest firm in M_i, for any final product $j \notin G^l$ and $j \in S_\alpha$, there does not exist $f_m \in L_\alpha$ such that $j \in G^m$ and $G^m \cap G^l \neq \varnothing$. This means that $G^l \subseteq S_\alpha$ forms a separable component, which contradicts with the assumption that S_α is a separable component. Therefore, when S_α is (**A1**), there always exists a firm $f_k \in L_\alpha$ such that $S_\alpha = G^k$.

Suppose that $f_k \in L_\alpha$ and $G^k = S_\alpha$ for an (**A1**) separable component $S_\alpha \in S$. Lemma 12 illustrates that $g^k(H)$ is differentiable on (H^0, ∞). We now show that $g^k(H)$ is strictly decreasing on (H^0, ∞). Assume for a contradiction that, $g^k(H^2) \geq g^k(H^1)$ for some $H^2 > H^1 > H^0$. With any given $H > H^0$, Lemma 12 shows that $\mu^k = g^k(H)$ satisfies

$$\frac{\mu^k - 1}{\mu^k} = \frac{\sum_{j \in G^k} \gamma_j(t_j(\Psi_j))}{H} \quad \text{by rearranging equation (14).} \tag{19}$$

Since $g^k(H^2) \geq g^k(H^1)$, we know that $\frac{g^k(H^2)-1}{g^k(H^2)} \geq \frac{g^k(H^1)-1}{g^k(H^1)}$, which implies that

$$\frac{\sum_{j \in G^k} \gamma_j(t_j(m_j(H^1)))}{H^1} \leq \frac{\sum_{j \in G^k} \gamma_j(t_j(m_j(H^2)))}{H^2} \quad \text{by equation (19).} \tag{20}$$

As the case where $\frac{\gamma_i(t_i(m_i(H^1)))}{H^1} = \frac{\gamma_i(t_i(m_i(H^2)))}{H^2}$ for every $i \in G^k$ clearly contradicts

[35] Note that S_α cannot be a strict subset of G^k, which violates the definition of a separable component.

equation (14) for firms in L_α under the assumption that $H^2 > H^1$, equation (20) implies that there exists at least a final product $i \in G^k$ such that $\frac{\gamma_i(t_i(m_i(H^1)))}{H^1} < \frac{\gamma_i(t_i(m_i(H^2)))}{H^2}$ and we denote such final products in G^k by $\hat{G}^k$. Moreover, for any final product $i \in \hat{G}^k$, $m_i(H^1) > m_i(H^2)$ by Lemmas 3 and 10 as $H^2 > H^1$. This further implies that there exists a firm $f_{l_i} \in M_i$ with $g^{l_i}(H^1) > g^{l_i}(H^2)$. By Lemma 12, $G^{l_i} \nsubseteq \hat{G}^k$, and

$$\frac{g^{l_i}(H^1) - 1}{g^{l_i}(H^1)} = \frac{\sum_{j \in G^{l_i}} \gamma_j(t_j(m_j(H^1)))}{H^1} > \frac{\sum_{j \in G^{l_i}} \gamma_j(t_j(m_j(H^2)))}{H^2} = \frac{g^{l_i}(H^2) - 1}{g^{l_i}(H^2)}.$$

Since S_α is (**A1**), for all $i, j \in \hat{G}^k$, either $G^{l_i} \subseteq G^{l_j}$ or $G^{l_j} \subseteq G^{l_i}$ when $G^{l_i} \cap G^{l_j} \neq \varnothing$. Then, we obtain

$$\hat{G}^k \subsetneq \bigcup_{j \in \hat{G}^k} G^{l_j} \equiv \Omega^k \subseteq G^k, \quad \text{and}$$

$$\frac{\sum_{j \in \Omega^k} \gamma_j(t_j(m_j(H^1)))}{H^1} > \frac{\sum_{j \in \Omega^k} \gamma_j(t_j(m_j(H^2)))}{H^2}.$$

This implies that

$$\frac{\sum_{j \in G^k} \gamma_j(t_j(m_j(H^1)))}{H^1} > \frac{\sum_{j \in G^k} \gamma_j(t_j(m_j(H^2)))}{H^2},$$

which contradicts inequality (20). It follows that $g^k(H)$ is strictly decreasing on (H^0, ∞). Taking the limit as H tends to infinity in equation (19), where $0 < \gamma_j(t_j(m_j(H))) < \gamma_j(t_j(1))$ for any $j \in G^k$ (by Lemma 12), shows that $\lim_{H \to \infty} g^k(H) = 1$. □

B.3 Proof of Proposition 3

Proof. For every $H > H^0$, Lemma 12 shows that $(\mu^k = g^k(H))_{f_k \in L_\alpha}$ satisfy

$$\mu^k = 1 + \mu^k \frac{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i))}{H}, \quad \forall f_k \in L_\alpha, \tag{21}$$

where $g^k(H) \in (1, \underline{\mu}^k)$ for any $f_k \in L_\alpha$ and $\Psi_i = m_i(H) \in (1, \bar{\mu}_i)$ for any $i \in S_\alpha$. This implies that $\gamma_i(t_i(\Psi_i)) > 0$ for any $i \in S_\alpha$ by Lemmas 3 and 10. Equation (21) is also equivalent to

$$\frac{\mu^k - 1}{\mu^k} \frac{1}{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i))} = \frac{1}{H}, \quad \forall f_k \in L_\alpha. \tag{22}$$

For every $H > H^0$ and $f_k \in L_\alpha$, let

$$\Phi^k(\mu^k) \equiv \frac{\mu^k - 1}{\mu^k} \frac{1}{\sum_{i \in G^k} \gamma_i(t_i(\mu^k))}, \tag{23}$$

where $\mu^k \in (1, \underline{\mu}^k)$. By Lemmas 3 and 10, $\Phi^k(\mu^k)$ is strictly increasing in μ^k on $(1, \underline{\mu}^k)$, and we obtain the following inequality for any $f_k \in L_\alpha$ with $(\mu^k = g^k(H))_{f_k \in L_\alpha}$:

$$\Phi^k(\mu^k) \leq \frac{\mu^k - 1}{\mu^k} \frac{1}{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i))} = \frac{1}{H}. \tag{24}$$

Define $\underline{H}^k \equiv \frac{1}{\Phi^k(g^k(H))}$ for any $H > H^0$. Note that, since S_α is either (**A1**) or (**A2**), every firm $f_k \in L_\alpha$ must have a final product $i \in G^k$ with $|M_i| \geq 2$, which implies that $\underline{H}^k > H$ for every $f_k \in L_\alpha$ by Lemmas 3 and 10. Since $\Phi^k(\mu^k)$ is strictly increasing in μ^k on $(1, \underline{\mu}^k)$, and $g^k(H) \in (1, \underline{\mu}^k)$, the equation $\Phi^k(\mu^k) = \frac{1}{\hat{H}}$ has a unique solution in $\mu^k \in (1, \underline{\mu}^k)$, denoted $\bar{g}^k(\hat{H})$, for any $\hat{H} \geq \underline{H}^k$, and $\bar{g}^k(\hat{H})$ is strictly decreasing in $\hat{H} \in (\underline{H}^k, \infty)$. Moreover, by equation (23) and inequality (24), $\bar{g}^k(\underline{H}^k) = g^k(H) \geq \bar{g}^k(\hat{H}) \geq g^k(\hat{H})$ for any $\hat{H} \geq \underline{H}^k > H$. Let $\underline{H} = \max_{f_k \in L_\alpha}(\underline{H}^k) > H$. Then, for any $H > H^0$ and $\hat{H} > \underline{H}$, we see that

$$g^k(H) \geq \bar{g}^k(\underline{H}) > \bar{g}^k(\hat{H}) \geq g^k(\hat{H}), \quad \forall f_k \in L_\alpha.$$

Since equation (21) with $(\mu^k = g^k(H))_{f_k \in L_\alpha}$ is equivalent to

$$\frac{\mu^k - 1}{\mu^k} = \frac{\sum_{i \in G^k} \gamma_i(t_i(\Psi_i))}{H},$$

for any $H > H^0$ and $f_k \in L_\alpha$, where $0 < \gamma_j(t_j(\Psi_j)) = \gamma_j(t_j(m_j(H))) < \gamma_j(t_j(1))$ for any $j \in G^k$ (by Lemma 12), taking the limit as H tends to infinity on both sides shows that $\lim_{H \to \infty} g^k(H) = 1$. □

B.4 Proof of Proposition 4

Proof. Note that Lemma 12 shows that $m_i(H)$ for every $i \in S_\alpha$ is differentiable on (H^0, ∞) with any network, and that, by Proposition 2, $m_i(H)$ is strictly decreasing on (H^0, ∞) for every $i \in S_\alpha$ in an (**S**) separable component. We now argue that in an (**A1**) separable component $S_\alpha \in S$, $m_i(H)$ is strictly decreasing on (H^0, ∞) for any $i \in S_\alpha$. We denote the firm providing intermediate products for all final products in S_α by f_k (i.e., $G^k = S_\alpha$). Lemma 2 shows that $g^k(H^2) < g^k(H^1)$ for any $H^2 > H^1 > H^0$. Assume for a contradiction that final product $i \in S_\alpha$ has the property that $m_i(H^2) \geq m_i(H^1)$ for some $H^2 > H^1 > H^0$. Since $f_k \in M_i$ and by Lemma 12, there exists at least a firm $f_m \in M_i$ with $g^m(H^2) > g^m(H^1)$ and $f_m \neq f_k$, and we denote the set of firms with this property in M_i by $\hat{M}_i$. Since S_α is (**A1**), we can pick the firm in $\hat{M}_i$, which provides intermediates for final products $\bigcap_{f_l \in \hat{M}_i} G^l$, and denote this firm by f_L (i.e., for

any firm $f_l \in \hat{M}_i$, $|G^L| \leq |G^l|$).[36] By Lemma 12, for every $H > H^0$, the η-markup for f_L, $\mu^L = g^L(H)$, satisfies

$$\frac{\mu^L - 1}{\mu^L} = \frac{\sum_{j \in G^L} \gamma_j(t_j(\Psi_j))}{H}. \tag{25}$$

Since $g^L(H^2) > g^L(H^1)$, we obtain

$$\frac{\sum_{j \in G^L} \gamma_j(t_j(m_j(H^2)))}{H^2} > \frac{\sum_{j \in G^L} \gamma_j(t_j(m_j(H^1)))}{H^1}. \tag{26}$$

Then, since, by Lemmas 3 and 10, $\gamma_i(t_i(m_i(H^2))) \leq \gamma_i(t_i(m_i(H^1)))$, there exists at least a final product $j \in G^L$ and $j \neq i$ such that $\gamma_j(t_j(m_j(H^2))) > \gamma_j(t_j(m_j(H^1)))$ and $M_i \neq M_j$, and we denote the set of such final products by $\hat{G}^L \subseteq G^L$. For some final product $j \in \hat{G}^L$, we argue that $g^m(H^2) \geq g^m(H^1)$ for every $f_m \in M_j$ with $G^m \subseteq G^L$. To see this, assume for a contradiction that for every $j \in \hat{G}^L$, there exists a firm $f_{x_j} \in M_j$ with $G^{x_j} \subseteq G^L$ such that $g^{x_j}(H^2) < g^{x_j}(H^1)$. By Lemma 12, $\mu^{x_j} = g^{x_j}(H)$ for every $H > H^0$ satisfies

$$\frac{\mu^{x_j} - 1}{\mu^{x_j}} = \frac{\sum_{y \in G^{x_j}} \gamma_y(t_y(m_y(H)))}{H},$$

which means that

$$\frac{\sum_{y \in G^{x_j}} \gamma_y(t_y(m_y(H^2)))}{H^2} < \frac{\sum_{y \in G^{x_j}} \gamma_y(t_y(m_y(H^1)))}{H^1}.$$

Then, since the network is (**A1**), either $G^{x_a} \subseteq G^{x_b}$ or $G^{x_b} \subseteq G^{x_a}$ if $G^{x_a} \bigcap G^{x_b} \neq \varnothing$ for any final products $a, b \in \hat{G}^L$. Note also that $\hat{G}^L \subseteq \bigcup_{j \in \hat{G}^L} G^{x_j} \subseteq G^L$ and $\gamma_q(t_q(m_q(H^2))) \leq \gamma_q(t_q(m_q(H^1)))$ for any final product $q \in G^L \setminus \hat{G}^L$. Hence, we see that

$$\frac{\sum_{y \in \bigcup_{j \in \hat{G}^L} G^{x_j}} \gamma_y(t_y(m_y(H^2)))}{H^2} < \frac{\sum_{y \in \bigcup_{j \in \hat{G}^L} G^{x_j}} \gamma_y(t_y(m_y(H^1)))}{H^1},$$

which implies that

$$\frac{\sum_{y \in G^L} \gamma_y(t_y(m_y(H^2)))}{H^2} < \frac{\sum_{y \in G^L} \gamma_y(t_y(m_y(H^1)))}{H^1}. \tag{27}$$

Inequality (27) contradicts inequality (26). It follows that we can pick a final product $j \in \hat{G}^L$ such that $g^m(H^2) \geq g^m(H^1)$ for every $f_m \in M_j$ with $G^m \subseteq G^L$. Since $\gamma_j(t_j(m_j(H^2))) > \gamma_j(t_j(m_j(H^1)))$ and by Lemmas 3 and 10, $m_j(H^2) < m_j(H^1)$. Note also that $m_i(H^2) \geq m_i(H^1)$. Then, by Lemma 12 and the conditions that $M_i \neq M_j$ and $g^m(H^2) \geq g^m(H^1)$ for every $f_m \in M_j$ with $G^m \subseteq G^L$, there exists another firm

[36] If multiple firms provide intermediates for $\bigcap_{f_l \in \hat{M}_i} G^l$, then f_L is an arbitrary one of these firms.

$f_z \in M_i$ and $f_z \notin M_j$ such that $g^z(H^2) > g^z(H^1)$. Since the network is (**A1**), this means that $G^z \subsetneq G^L$ and $|G^z| < |G^L|$, which contradicts the property that $|G^l| \geq |G^L|$ for every $f_l \in \hat{M}_i$. Hence, $m_i(H^2) < m_i(H^1)$ for every final product $i \in S_\alpha$ and every $H^2 > H^1 > H^0$.

Proposition 3 shows that $\lim_{H\to\infty} g^l(H) = 1$ for any $f_l \in L_\alpha$, which implies that, by Lemma 12, $\lim_{H\to\infty} m_i(H) = \lim_{H\to\infty} \sum_{f_l \in M_i} g^l(H) = |M_i|$ for any $i \in S_\alpha$. □

B.5 Proof of Proposition 5

Proof. We first show that the lowest and highest equilibrium aggregators exist. By Lemma 13 and its proof, the set of equilibrium aggregators is nonempty, bounded (since it lies in $[H^0, H^0 + \sum_{i\in\mathcal{N}} h_i(C_i)]$ with $H^0 > \underline{H}^0 \geq 0$), and closed (since it is the set $\{H \geq H^0 : \Gamma(H) - H = 0\}$ and $\Gamma(H) - H$ is a continuous function on $[H^0, \infty)$). Hence, the set of equilibrium aggregators is compact, and its minimum and maximum exist (see also the proof of Proposition 3 in Nocke and Schutz (2018), which follows the same argument).

Define $\hat{\Gamma}(H, H^0) \equiv \Gamma(H, H^0) - H$,[37] where

$$\Gamma(H, H^0) - H = H^0 + \sum_{i\in\mathcal{N}} h_i(t_i(m_i(H))) - H.$$

Note that by Lemma 13, $H > H^0$ is an equilibrium aggregator if and only if $\hat{\Gamma}(H, H^0) = 0$. We now argue that as the value of H^0 increases, consumer surplus increases under any network in the equilibria with the lowest and highest aggregator H. To see this, we need to show some key properties of the function $\hat{\Gamma}(H, H^0)$ with $H \geq H^0$. We know from the proof of Lemma 13 that $\Gamma(H, H^0) \in (H^0, H^0 + \sum_{i\in\mathcal{N}} h_i(C_i))$ for every $H \geq H^0$. This means that $\hat{\Gamma}(H, H^0) > 0$ when $H \leq H^0$ and $\hat{\Gamma}(H, H^0) < 0$ when $H \geq H^0 + \sum_{i\in\mathcal{N}} h_i(C_i)$. Moreover, by Lemma 12, $\hat{\Gamma}(H, H^0)$ is continuous on $[H^0, \infty)$. Let $\underline{H}^*$ be the lowest equilibrium aggregator when the outside option is H^0, which means that $\underline{H}^*$ is the lowest value of H such that $\hat{\Gamma}(H, H^0) = 0$. Then, since $\hat{\Gamma}(H, H^0) > 0$ when $H = H^0$ and $\hat{\Gamma}(H, H^0)$ is continuous on $[H^0, \infty)$, $\hat{\Gamma}(H, H^0) \geq 0$ for any $H \leq \underline{H}^*$. Next, let $\underline{H}^{*'}$ be the lowest equilibrium aggregator when the outside option is $H^{0'}$. Suppose that $H^{0'} > H^0$. We argue that $\underline{H}^{*'} > \underline{H}^*$. Clearly, $\hat{\Gamma}(H, H^{0'}) > \hat{\Gamma}(H, H^0)$ for any $H \geq H^0$. Then, we have

$$\hat{\Gamma}(H, H^{0'}) > \hat{\Gamma}(H, H^0) \geq 0, \quad \forall H \leq \underline{H}^*.$$

[37] Since H^0 varies in the remainder of this proof, we write $\Gamma(H, H^0)$ for the aggregate fitting-in function to make its dependence on H^0 explicit. Note that the fitting-in functions $(m_i)_{i\in\mathcal{N}}$ are determined by equations (6) for every $H > \underline{H}^0$, so H^0 only affects their domain. Hence, H^0 enters $\Gamma(H, H^0)$ only through the additive term H^0 and through the domain $H \geq H^0$.

Therefore, when the outside option is $H^{0'}$, there is no equilibrium aggregator on $(H^0, \underline{H}^*]$, which shows that $\underline{H}^{*'} > \underline{H}^*$. Similarly, we can show that $\overline{H}^* < \overline{H}^{*'}$, where $\overline{H}^*$ and $\overline{H}^{*'}$ are the highest equilibrium aggregators when the outside options are H^0 and $H^{0'}$, respectively.

The remaining parts of the proposition follow from Propositions 2 and 4 and Lemmas 2 and 10. The effects of an entry of a new firm providing substitutable final products for all the incumbent firms follow from the same line of argument. This firm forms a separable component, and with any $H > H^0$, the entry shifts $\hat{\Gamma}(H, H^0)$ upward.[38] □

B.6 Proof of Example 1

Proof. This (**A1**) network is covered by Proposition 6. In the proof of Proposition 6, we study how a cost reduction may affect the pricing functions and the fitting-in functions and show that this cost reduction can always increase equilibrium consumer surplus. We will use these results and the notations in the proof of Proposition 6 to prove Example 1.

Suppose that $C_1 > C_1' > 0$. Under this MNL specification, we know that, by Lemma 10, $t_1(\Psi_1) = \Psi_1 + C_1 > \hat{t}_1(\Psi_1) = \Psi_1 + C_1'$ for every $\Psi_1 \in (1, \infty)$ and $t_2(\Psi_2) = \Psi_2 + C_2$ for every $\Psi_2 \in (1, \infty)$. Since $H^* < \hat{H}^*$ and $\frac{\Gamma(H^*)}{H^*} = \frac{\hat{\Gamma}(\hat{H}^*)}{\hat{H}^*} = 1$, it follows that

$$\frac{h_1(m_1(H^*) + C_1) + h_2(m_2(H^*) + C_2)}{H^*} < \frac{h_1(\hat{m}_1(\hat{H}^*) + C_1') + h_2(\hat{m}_2(\hat{H}^*) + C_2)}{\hat{H}^*}, \quad (28)$$

where $m_1(H^*) = g^1(H^*) + g^2(H^*)$, $m_2(H^*) = g^1(H^*) + g^3(H^*)$, $\hat{m}_1(\hat{H}^*) = \hat{g}^1(\hat{H}^*) + \hat{g}^2(\hat{H}^*)$, and $\hat{m}_2(\hat{H}^*) = \hat{g}^1(\hat{H}^*) + \hat{g}^3(\hat{H}^*)$. By equation (35) for firm f_1, this implies that $g^1(H^*) < \hat{g}^1(\hat{H}^*)$. Then, by inspecting equation (35) for firm f_3, where $h_2' < 0$ on $(0, \infty)$, we see that the function $\tau^3(x, y, H) = \frac{x-1}{x} - \frac{h_2(y+x+C_2)}{H}$ is strictly increasing in $x \in (1, \infty)$, in $y \in (1, \infty)$, and in $H \in (H^0, \infty)$. By $\tau^3(g^3(H^*), g^1(H^*), H^*) = \tau^3(\hat{g}^3(\hat{H}^*), \hat{g}^1(\hat{H}^*), \hat{H}^*) = 0$ from equation (35) and the properties that $g^1(H^*) < \hat{g}^1(\hat{H}^*)$ and $H^* < \hat{H}^*$, we obtain

$$\frac{g^3(H^*) - 1}{g^3(H^*)} = \frac{h_2(g^1(H^*) + g^3(H^*) + C_2)}{H^*} > \frac{\hat{g}^3(\hat{H}^*) - 1}{\hat{g}^3(\hat{H}^*)} = \frac{h_2(\hat{g}^1(\hat{H}^*) + \hat{g}^3(\hat{H}^*) + C_2)}{\hat{H}^*},$$

which means that $g^3(H^*) > \hat{g}^3(\hat{H}^*)$. Hence, by equation (28),

$$\frac{h_1(m_1(H^*) + C_1)}{H^*} < \frac{h_1(\hat{m}_1(\hat{H}^*) + C_1')}{\hat{H}^*}. \quad (29)$$

[38] Let the new firm be f_l. Then, this firm shifts $\hat{\Gamma}(H, H^0)$ upward by at least $\sum_{i \in G^l} h_i(t_i(m_i(H^0))) > 0$.

Since $H^* < \hat{H}^*$ and $h_1' < 0$, inequality (29) implies that

$$\begin{aligned} &h_1(m_1(H^*) + C_1) < h_1(\hat{m}_1(\hat{H}^*) + C_1'), \quad \text{and} \\ &t_1(m_1(H^*)) = m_1(H^*) + C_1 > \hat{t}_1(\hat{m}_1(\hat{H}^*)) = \hat{m}_1(\hat{H}^*) + C_1'. \end{aligned} \tag{30}$$

It means that the equilibrium price of final product 1 gets lower after the cost reduction.

We now study the impact of the cost reduction on the price of final product 2. Using equations (35) for firms f_1 and f_3 at H^* and $\hat{H}^*$, where $\frac{\Gamma(H^*)}{H^*} = \frac{\hat{\Gamma}(\hat{H}^*)}{\hat{H}^*} = 1$, we obtain

$$\begin{aligned} m_2(H^*) &= 2 + \frac{h_1(m_1(H^*) + C_1) + h_2(m_2(H^*) + C_2)}{H^0} + \frac{h_2(m_2(H^*) + C_2)}{H^0 + h_1(m_1(H^*) + C_1)}, \\ \hat{m}_2(\hat{H}^*) &= 2 + \frac{h_1(\hat{m}_1(\hat{H}^*) + C_1') + h_2(\hat{m}_2(\hat{H}^*) + C_2)}{H^0} + \frac{h_2(\hat{m}_2(\hat{H}^*) + C_2)}{H^0 + h_1(\hat{m}_1(\hat{H}^*) + C_1')}. \end{aligned} \tag{31}$$

Define

$$\Theta(x, y) = 2 + \frac{x + h_2(y + C_2)}{H^0} + \frac{h_2(y + C_2)}{H^0 + x} - y,$$

where $x \in (0, \infty)$ and $y \in (1, \infty)$. This function has the properties that $\frac{\partial \Theta}{\partial y} < 0$, and $\frac{\partial \Theta}{\partial x} > 0$ if and only if $x > \sqrt{H^0 h_2(y + C_2)} - H^0$. Under condition (a), equations (35) for firms f_1, f_2, and f_3 show that $m_1(H^*) = m_2(H^*)$ and $h_1(m_1(H^*) + C_1) = h_2(m_2(H^*) + C_2)$. Then, we know that $h_1(m_1(H^*) + C_1) > \sqrt{H^0 h_2(m_2(H^*) + C_2)} - H^0$ for any $H^0 > 0$. By inequality (30) and the properties of the function $\Theta(x, y)$, this implies that

$$\begin{aligned} 0 &= 2 + \frac{h_1(m_1(H^*) + C_1) + h_2(m_2(H^*) + C_2)}{H^0} + \frac{h_2(m_2(H^*) + C_2)}{H^0 + h_1(m_1(H^*) + C_1)} - m_2(H^*), \\ &< 2 + \frac{h_1(\hat{m}_1(\hat{H}^*) + C_1') + h_2(m_2(H^*) + C_2)}{H^0} + \frac{h_2(m_2(H^*) + C_2)}{H^0 + h_1(\hat{m}_1(\hat{H}^*) + C_1')} - m_2(H^*). \end{aligned} \tag{32}$$

Since $\frac{\partial \Theta}{\partial y} < 0$, equation (31) and inequality (32) show that $m_2(H^*) + C_2 < \hat{m}_2(\hat{H}^*) + C_2$, which means that the equilibrium price of final product 2 gets higher after the cost reduction.

We now consider the condition that $H^0 > h_2(m_2(H^*) + C_2) > 0$.[39] This implies that $\sqrt{H^0 h_2(m_2(H^*) + C_2)} - H^0 < 0$. Then, since $h_1 > 0$ and by Lemma 12, $h_1(m_1(H^*) + C_1) > 0 > \sqrt{H^0 h_2(m_2(H^*) + C_2)} - H^0$. By the same argument as in the proof with condition (a), the cost reduction increases the equilibrium price of final product 2. □

[39] Note that this condition can be easily satisfied. For example, since $m_2(H^*) > 1$ and $h_2' < 0$, $H^0 > h_2(m_2(H^*) + C_2)$ when $H^0 \geq h_2(1 + C_2)$.

B.7 Proof of Proposition 6

Proof. Under any given network with the demand system $((h_j)_{j\in\mathcal{N}}, H^0)$, we first evaluate how a reduction in C_i, the total marginal cost of final product $i \in \mathcal{N}$, affects the pricing function, $t_i(\Psi_i)$, for final product i. Suppose that $C_i > C_i' > 0$. Let $t_i(\Psi_i)$ (resp. $\hat{t}_i(\Psi_i)$) be the pricing function for final product i when the total marginal cost of final product i is C_i (resp. C_i'). Since $t_i(\Psi_i)$ is the unique solution of $\frac{P_i - C_i}{P_i}\eta_i(P_i) = \Psi_i$ with $\Psi_i \in (1, \bar{\mu}_i)$ and $P_i \in (0, \infty)$, as shown in Lemma 10, we see that for every $\Psi_i \in (1, \bar{\mu}_i)$,

$$t_i(\Psi_i) > \hat{t}_i(\Psi_i). \tag{33}$$

Also, it is obvious that $t_j(\Psi_j)$ for any $j \in \mathcal{N} \setminus \{i\}$ is not affected by changes in C_i.

We now study how a reduction in C_i affects firms' and final products' fitting-in functions with CES or MNL demands in an (**S**) or (**A1**) separable component. Assumption 2a is always satisfied so that the fitting-in functions are well-defined.[40] In the CES case, let $h_j(P_j) = a_j P_j^{1-\sigma}$ for every $j \in \mathcal{N}$, where $a_j > 0$ and $\sigma > 1$. In the MNL case, let $h_j(P_j) = \exp(\frac{a_j - P_j}{\lambda})$ for every $j \in \mathcal{N}$, where $a_j \in \mathbb{R}$ and $\lambda > 0$. Note that by Lemma 3, for every $j \in \mathcal{N}$, we have

$$\gamma_j(P_j) = \begin{cases} a_j \frac{\sigma-1}{\sigma} P_j^{1-\sigma} = \frac{\sigma-1}{\sigma} h_j(P_j) & \text{(CES)}, \\ \exp(\frac{a_j - P_j}{\lambda}) = h_j(P_j) & \text{(MNL)}. \end{cases} \tag{34}$$

Suppose that final product $i \in S_\alpha$ and $S_\alpha \in S$. By Lemma 12, when final products' total marginal costs are $(C_i)_{i\in\mathcal{N}}$, the outputs of fitting-in functions $(\mu^k = g^k(H))_{f_k \in L_\alpha}$ and $(\Psi_j = m_j(H))_{j\in S_\alpha}$ with $H \in (H^0, \infty)$ are uniquely determined by a system of equations (14) for all firms in L_α, which can be rewritten as follows:

$$\frac{\mu^k - 1}{\mu^k} = \frac{\sum_{q\in G^k} \gamma_q(t_q(\Psi_q))}{H}, \quad \forall f_k \in L_\alpha. \tag{35}$$

Let $\hat{g}^k(H)$ and $\hat{m}_j(H)$ be the fitting-in functions for any firm $f_k \in L_\alpha$ and any final product $j \in S_\alpha$, respectively, when the total marginal cost of final product i is changed to C_i'. By Lemma 12, fitting-in functions for firms and final products outside S_α will not be affected.

We first show that when S_α is (**S**) or (**A1**), there exists a firm $f_l \in L_\alpha$ such that $\hat{g}^l(H) > g^l(H)$ for any $H > H^0$. Assume for a contradiction that $\hat{g}^k(H) \leq g^k(H)$ for any $f_k \in L_\alpha$ at some $H = H' > H^0$. By Lemma 12, we know that $\hat{m}_j(H') \leq m_j(H')$ for

[40] Let $\underline{H}^{0'}$ and $\underline{H}^{0''}$ be the lower bounds of H^0, defined in the proof of Theorem 1, before and after the cost reduction in final product $i \in \mathcal{N}$ with a CES demand specification. To ensure that the fitting-in functions are well-defined and the impact only comes from the cost reduction, we assume that $\underline{H}^0 = \max(\underline{H}^{0'}, \underline{H}^{0''})$ before and after the cost reduction.

any $j \in S_\alpha$. Then, since $t_i(\Psi_i) > \hat{t}_i(\Psi_i)$ for every $\Psi_i \in (1, \bar{\mu}_i)$ and by Lemmas 3 and 10, we get $t_i(m_i(H')) > \hat{t}_i(\hat{m}_i(H'))$ and $\gamma_i(\hat{t}_i(\hat{m}_i(H'))) > \gamma_i(t_i(m_i(H')))$. Similarly, for any $j \in S_\alpha \setminus \{i\}$, we obtain $\gamma_j(t_j(\hat{m}_j(H'))) \geq \gamma_j(t_j(m_j(H')))$. This means that, by equation (35), for every $f_m \in M_i$, where $M_i \subseteq L_\alpha$, we obtain

$$\underbrace{\frac{\gamma_i(\hat{t}_i(\hat{m}_i(H')))}{H'} + \frac{\sum_{q \in G^m \setminus \{i\}} \gamma_q(t_q(\hat{m}_q(H')))}{H'}}_{= \frac{\hat{g}^m(H')-1}{\hat{g}^m(H')}} > \underbrace{\frac{\sum_{q \in G^m} \gamma_q(t_q(m_q(H')))}{H'}}_{= \frac{g^m(H')-1}{g^m(H')}},$$

which implies that $\hat{g}^m(H') > g^m(H')$. This contradicts the assumption that $\hat{g}^k(H') \leq g^k(H')$ for any $f_k \in L_\alpha$. Hence, when S_α is $(\mathbf{S})$ or $(\mathbf{A1})$ with $C_i > C_i' > 0$, there exists a firm $f_l \in L_\alpha$ such that $\hat{g}^l(H) > g^l(H)$ for any $H > H^0$.

Next, we study how a cost reduction affects the aggregate fitting-in function when S_α is $(\mathbf{S})$. Then, $G^k = G^m = S_\alpha$ for any $f_k, f_m \in L_\alpha$, which implies that, by equation (35), $g^k(H) = g^m(H)$ and $\hat{g}^k(H) = \hat{g}^m(H)$ for any $f_k, f_m \in L_\alpha$ and $H > H^0$. Moreover, since there must exist a firm $f_l \in L_\alpha$ such that $\hat{g}^l(H) > g^l(H)$ for every $H > H^0$, it follows that $\hat{g}^k(H) > g^k(H)$ for every $f_k \in L_\alpha$ and $H > H^0$. By equations (34) and (35), we know that for every $H > H^0$,

$$\frac{h_i(\hat{t}_i(\hat{m}_i(H)))}{H} + \frac{\sum_{q \in S_\alpha \setminus \{i\}} h_q(t_q(\hat{m}_q(H)))}{H} > \frac{\sum_{q \in S_\alpha} h_q(t_q(m_q(H)))}{H}. \tag{36}$$

Let $\Gamma(H)$ (resp. $\hat{\Gamma}(H)$) be the aggregate fitting-in function, defined in Lemma 13, when the total marginal cost of final product i is C_i (resp. C_i'). Since fitting-in functions for firms and final products outside S_α are not affected by C_i or C_i' with every $H > H^0$, equation (34) and inequality (36) show that $\frac{\hat{\Gamma}(H)}{H} > \frac{\Gamma(H)}{H}$ for every $H > H^0$.

We now assume that S_α is $(\mathbf{A1})$. As shown in the proof of Lemma 2, there exists a firm in L_α, denoted f_k, such that $G^k = S_\alpha$. Note that if $\hat{g}^k(H) > g^k(H)$ for every $H > H^0$, then, by equations (34) and (35),

$$\frac{h_i(\hat{t}_i(\hat{m}_i(H)))}{H} + \frac{\sum_{q \in S_\alpha \setminus \{i\}} h_q(t_q(\hat{m}_q(H)))}{H} > \frac{\sum_{q \in S_\alpha} h_q(t_q(m_q(H)))}{H}.$$

By the same argument for the case when S_α is $(\mathbf{S})$, we obtain $\frac{\hat{\Gamma}(H)}{H} > \frac{\Gamma(H)}{H}$ for every $H > H^0$. Hence, all we need to do now is show that $\hat{g}^k(H) > g^k(H)$ for every $H > H^0$. Assume for a contradiction that $\hat{g}^k(H) \leq g^k(H)$ at some $H = H' > H^0$. Since we have shown that there exists a firm $f_l \in L_\alpha$ such that $\hat{g}^l(H') > g^l(H')$, it follows that $G^l \subsetneq G^k$. Then, by equation (35) and the assumption that $\hat{g}^k(H') \leq g^k(H')$, there exists at least a final product $j \in G^k \setminus G^l$, such that $\gamma_j(t_j(\hat{m}_j(H'))) < \gamma_j(t_j(m_j(H')))$ if $j \neq i$ and $\gamma_j(\hat{t}_j(\hat{m}_j(H'))) < \gamma_j(t_j(m_j(H')))$ if $j = i$, and we denote the set of such final products by $\underline{G}^k$. Moreover, by Lemmas 3 and 10, for any final product $j \in \underline{G}^k$,

$\hat{m}_j(H') = \sum_{f_m \in M_j} \hat{g}^m(H') > m_j(H') = \sum_{f_m \in M_j} g^m(H')$, which means that there exists a firm f_y in M_j such that $\hat{g}^y(H') > g^y(H')$. Since the network is (**A1**), where any two firms providing complementary products have nested product sets, we can collect a set of firms, denoted $\mathcal{T}^k$, providing intermediates for $\underline{G}^k$, such that $G^a \cap G^b = \varnothing$ for any $f_a, f_b \in \mathcal{T}^k$, $\underline{G}^k \subseteq \bigcup_{f_m \in \mathcal{T}^k} G^m$, and $\hat{g}^m(H') > g^m(H')$ for every $f_m \in \mathcal{T}^k$. Then, by equation (35), we see that

$$\sum_{f_m \in \mathcal{T}^k} \frac{g^m(H') - 1}{g^m(H')} = \frac{\sum_{q \in (\bigcup_{f_m \in \mathcal{T}^k} G^m)} \gamma_q(t_q(m_q(H')))}{H'}, \quad \text{and}$$

$$\sum_{f_m \in \mathcal{T}^k} \frac{\hat{g}^m(H') - 1}{\hat{g}^m(H')} = \begin{cases} \frac{\sum_{q \in (\bigcup_{f_m \in \mathcal{T}^k} G^m)} \gamma_q(t_q(\hat{m}_q(H')))}{H'} & \text{if } i \notin \bigcup_{f_m \in \mathcal{T}^k} G^m, \\ \frac{\sum_{q \in (\bigcup_{f_m \in \mathcal{T}^k} G^m) \setminus \{i\}} \gamma_q(t_q(m_q(H')))}{H'} + \frac{\gamma_i(\hat{t}_i(\hat{m}_i(H')))}{H'} & \text{if } i \in \bigcup_{f_m \in \mathcal{T}^k} G^m, \end{cases}$$

where $\sum_{f_m \in \mathcal{T}^k} \frac{g^m(H')-1}{g^m(H')} < \sum_{f_m \in \mathcal{T}^k} \frac{\hat{g}^m(H')-1}{\hat{g}^m(H')}$. Since $\gamma_j(t_j(\hat{m}_j(H'))) \geq \gamma_j(t_j(m_j(H')))$ for any $j \in G^k \setminus (\underline{G}^k \cup G^l)$ and $j \neq i$, $\gamma_i(\hat{t}_i(\hat{m}_i(H'))) \geq \gamma_i(t_i(m_i(H')))$ if $i \in G^k \setminus (\underline{G}^k \cup G^l)$, and $(\bigcup_{f_m \in \mathcal{T}^k} G^m) \cap G^l = \varnothing$ (or G^l, as each G^m for firm $f_m \in \mathcal{T}^k$ contains a final product in $\underline{G}^k \subseteq G^k \setminus G^l$), it follows that

$$\frac{h_i(\hat{t}_i(\hat{m}_i(H')))}{H'} + \frac{\sum_{q \in G^k \setminus \{i\}} h_q(t_q(\hat{m}_q(H')))}{H'} > \frac{\sum_{q \in G^k} h_q(t_q(m_q(H')))}{H'}.$$

By equation (35), this contradicts the assumption that $\hat{g}^k(H') \leq g^k(H')$. Therefore, $\hat{g}^k(H) > g^k(H)$ for every $H > H^0$. By the same argument for the case when S_α is (**S**), we get $\frac{\hat{\Gamma}(H)}{H} > \frac{\Gamma(H)}{H}$ for every $H > H^0$.

As shown later in the proofs of Propositions 8 and 9, uniqueness results independent of this section, $\frac{\Gamma(H)}{H}$ and $\frac{\hat{\Gamma}(H)}{H}$ are strictly decreasing in H on (H^0, ∞) when the network is (**S**) or (**A1**). Then, the result that $\frac{\hat{\Gamma}(H)}{H} > \frac{\Gamma(H)}{H}$ for every $H > H^0$ in an (**S**) or (**A1**) network implies that $H^* < \hat{H}^*$, where $H^* \in (H^0, \infty)$ is the unique solution of $\frac{\Gamma(H)}{H} = 1$ and $\hat{H}^* \in (H^0, \infty)$ is the unique solution of $\frac{\hat{\Gamma}(H)}{H} = 1$. Since H^* and $\hat{H}^*$ correspond to two equilibrium aggregators by Lemma 13, it follows that by Proposition 1, equilibrium consumer surplus is improved after a reduction in C_i in an (**S**) or (**A1**) network. □

B.8 Proof of Proposition 7

Proof. Let $h_j(P_j) = \exp(a - P_j) = h(P_j)$ for every $j \in \mathcal{N}$, where $a \in \mathbb{R}$. This MNL demand satisfies Assumption 2b with any $H^0 > 0$. Note that by Lemma 3, for every $j \in \mathcal{N}$, $\gamma_j(P_j) = h_j(P_j)$. Let $0 < C_1' < C_1 = C_2 = C_3 = C$ and $\Delta C_1 = C - C_1'$. By Lemma 10, we know that $t_1(\Psi_1) = \Psi_1 + C > \hat{t}_1(\Psi_1) = \Psi_1 + C_1'$ for every $\Psi_1 \in (1, \infty)$, where

$t_1(\Psi_1)$ (resp. $\hat{t}_1(\Psi_1)$) is the pricing function for final product 1 when the total marginal cost of final product 1 is C (resp. C_1'). For final products 2 and 3, $t_2(\Psi_2) = \Psi_2 + C$ with $\Psi_2 \in (1, \infty)$ and $t_3(\Psi_3) = \Psi_3 + C$ with $\Psi_3 \in (1, \infty)$.

We now evaluate how a reduction in C_1 affects firms' and final products' fitting-in functions under this (**A2**) network. By Lemma 12, when the total marginal cost of each final product is C, the outputs of fitting-in functions $(\mu^k = g^k(H))_{f_k \in \mathcal{F}}$ and $(\Psi_j = m_j(H))_{j \in \mathcal{N}}$ with $H \in (H^0, \infty)$ solve the following equations:

$$\begin{aligned} \frac{\mu^1 - 1}{\mu^1} &= \frac{\gamma_1(t_1(\Psi_1)) + \gamma_2(t_2(\Psi_2))}{H} = \frac{h_1((\mu^1 + \mu^2) + C) + h_2(\mu^1 + C)}{H}, \\ \frac{\mu^2 - 1}{\mu^2} &= \frac{\gamma_1(t_1(\Psi_1)) + \gamma_3(t_3(\Psi_3))}{H} = \frac{h_1((\mu^1 + \mu^2) + C) + h_3(\mu^2 + C)}{H}. \end{aligned} \tag{37}$$

Let $\hat{g}^k(H)$ and $\hat{m}_j(H)$ be the fitting-in functions for any firm $f_k \in \mathcal{F}$ and any final product $j \in \mathcal{N}$, respectively, when the total marginal cost of final product 1 is C_1'. Since $h_2(P) = h_3(P)$ for any $P > 0$ and by inspecting equation (37), $g^1(H) = g^2(H) \equiv g(H)$ and $\hat{g}^1(H) = \hat{g}^2(H) \equiv \hat{g}(H)$ for every $H > H^0$. Then, equation (37) with $\mu = g(H)$, $C_1 = C$ and $H > H^0$ can be rewritten as

$$\frac{\mu - 1}{\mu} - \frac{h(2\mu + C) + h(\mu + C)}{H} = 0. \tag{38}$$

Similarly, by Lemma 12, when $\mu' = \hat{g}(H)$, $H > H^0$, and the total marginal cost of final product 1 is C_1', we obtain

$$\frac{\mu' - 1}{\mu'} - \frac{h(2\mu' + C_1') + h(\mu' + C)}{H} = 0. \tag{39}$$

We first show that $\hat{g}(H) > g(H)$ for every $H > H^0$. By Lemma 3, the left-hand side of equation (38) is strictly increasing in $\mu \in (1, \infty)$ and strictly increasing in $C \in (0, \infty)$. Since the left-hand side of equation (39) is essentially the left-hand side of equation (38) by setting $C_1 = C_1' < C$, we can conclude that $\hat{g}(H) > g(H)$ for any $H > H^0$. Moreover, since the left-hand side of equation (38) is strictly increasing in $\mu \in (1, \infty)$ and in $H \in (0, \infty)$, we see that $g(H)$ is strictly decreasing in H on (H^0, ∞). Similarly, by equation (39), $\hat{g}(H)$ is strictly decreasing in H on (H^0, ∞). These results also show that $\hat{m}_j(H) > m_j(H)$ for any $j \in \mathcal{N}$ and $H > H^0$, and both $\hat{m}_j(H)$ and $m_j(H)$ are strictly decreasing in H on (H^0, ∞).

Next, we study how a decrease in C_1 affects the aggregate fitting-in function, which determines the equilibrium aggregator. Let $\Gamma(H)$ (resp. $\hat{\Gamma}(H)$) be the aggregate fitting-in function on (H^0, ∞), defined in Lemma 13, when the total marginal cost of final product 1 is C (resp. C_1'). As shown later in the proof of Proposition 8, a uniqueness

result independent of this proof, $\frac{\Gamma(H)}{H}$ and $\frac{\hat{\Gamma}(H)}{H}$ are strictly decreasing in H on (H^0, ∞), which leads to a unique solution of $\frac{\Gamma(H)}{H} = 1$ on (H^0, ∞), denoted H^*, and a unique solution of $\frac{\hat{\Gamma}(H)}{H} = 1$ on (H^0, ∞), denoted $\hat{H}^*$. By Lemma 13, H^* and $\hat{H}^*$ are the equilibrium aggregators when $C_1 = C$ and $C_1 = C_1'$, respectively.

We first prove a necessary and sufficient condition for $\hat{H}^* < H^*$. Since $C_1' < C$ and $g(H^*) > 1$, and by Lemma 3, there exists a unique $\beta \in (1, 1 + \Delta C_1)$ such that

$$\Gamma(H^*) - H^0 = h(2g(H^*) + C) + 2h(g(H^*) + C) = h(2\beta g(H^*) + C_1') + 2h(\beta g(H^*) + C). \tag{40}$$

We argue that $\hat{g}(H^*) > \beta g(H^*)$, which is equivalent to $\hat{H}^* < H^*$, if and only if

$$\frac{\beta g(H^*) - 1}{\beta g(H^*)} < \frac{h(2\beta g(H^*) + C_1') + h(\beta g(H^*) + C)}{H^*}. \tag{41}$$

To see this, inequality (41) implies that

$$\frac{\beta g(H^*) - 1}{\beta g(H^*)} - \frac{h(2\beta g(H^*) + C_1') + h(\beta g(H^*) + C)}{H^*} < 0. \tag{42}$$

Let

$$\phi(x, H^*) = \frac{x - 1}{x} - \frac{h(2x + C_1') + h(x + C)}{H^*}.$$

By Lemma 3, $\phi(x, H^*)$ is strictly increasing in x on $(1, \infty)$, and it is obvious that $\phi(x, H^*) = 0$ has a unique solution in x on $(1, \infty)$, denoted $x^*(H^*)$. By inequality (42), we obtain $x^*(H^*) > \beta g(H^*)$. Since, by equation (39),

$$\frac{\hat{g}(H^*) - 1}{\hat{g}(H^*)} - \frac{h(2\hat{g}(H^*) + C_1') + h(\hat{g}(H^*) + C)}{H^*} = 0.$$

It follows that $\hat{g}(H^*) = x^*(H^*) > \beta g(H^*)$. Then, by equation (40) and Lemma 3,

$$\begin{aligned} 1 = \frac{\Gamma(H^*)}{H^*} &= \frac{h(2g(H^*) + C) + 2h(g(H^*) + C) + H^0}{H^*} \\ > \frac{\hat{\Gamma}(H^*)}{H^*} &= \frac{h(2\hat{g}(H^*) + C_1') + 2h(\hat{g}(H^*) + C) + H^0}{H^*}. \end{aligned}$$

Therefore, when $H = H^*$, $\frac{\hat{\Gamma}(H^*)}{H^*} < 1$. Since $\frac{\hat{\Gamma}(H)}{H}$ is strictly decreasing in H on (H^0, ∞), we obtain $\hat{H}^* < H^*$. By reversing the steps above, we can prove that the condition is also necessary for $\hat{H}^* < H^*$.

Then, we use this condition to show that if $H^0 < \min(h(m_1(H^*) + C), \tilde{H}^0)$, where $\tilde{H}^0 = h(g(H^*) + C)(\frac{1}{\Delta C_1} - \frac{1 + \Delta C_1}{\Delta C_1 \exp(g(H^*)\Delta C_1)})$, then $\hat{H}^* < H^*$. By rearranging equation

(38) and using the condition that $\frac{\Gamma(H^*)}{H^*} = 1$, we know that

$$\frac{g(H^*)-1}{g(H^*)} = \frac{h(2g(H^*)+C)+h(g(H^*)+C)}{h(2g(H^*)+C)+2h(g(H^*)+C)+H^0} < \frac{h(2g(H^*)+C)+h(g(H^*)+C)}{h(2g(H^*)+C)+2h(g(H^*)+C)}, \tag{43}$$

which implies that $\exp(-g(H^*)) > g(H^*)-2$ and $g(H^*) < 2.13$. When $H^0 < h(2g(H^*)+C)$,

$$\frac{g(H^*)-1}{g(H^*)} = \frac{h(2g(H^*)+C)+h(g(H^*)+C)}{h(2g(H^*)+C)+2h(g(H^*)+C)+H^0} > \frac{h(2g(H^*)+C)+h(g(H^*)+C)}{2(h(2g(H^*)+C)+h(g(H^*)+C))} = \frac{1}{2}.$$

Hence, if $H^0 < h(2g(H^*)+C)$, $2 < g(H^*) < 2.13$. This also shows that $H^0 < h(5+C)$ always ensures that $H^0 < h(2g(H^*)+C)$. Since $\frac{1}{x} - \frac{1+x}{x\exp(xg(H^*))}$ is strictly decreasing in x on $(0,\infty)$ for any $g(H^*) > 2$ and since $0 < \beta - 1 < \Delta C_1$, the assumption $H^0 < \min(h(m_1(H^*)+C), \tilde{H}^0)$ also implies that

$$\begin{aligned} H^0 &< h(g(H^*)+C)\left(\frac{1}{\Delta C_1} - \frac{1+\Delta C_1}{\Delta C_1 \exp(g(H^*)\Delta C_1)}\right) \\ &< h(g(H^*)+C)\left(\frac{1}{\beta-1} - \frac{\beta}{(\beta-1)\exp\left((\beta-1)g(H^*)\right)}\right) \\ &= \frac{h(g(H^*)+C) - \beta h(\beta g(H^*)+C)}{\beta-1}. \end{aligned} \tag{44}$$

Note that, since $1 < \beta < 1+\Delta C_1$ and $1 < g(H^*) < \infty$ for any $H^0 > 0$ by Lemmas 12 and 13,

$$\frac{\beta h(\beta g(H^*)+C)}{h(g(H^*)+C)} = \frac{\beta}{\exp((\beta-1)g(H^*))} < 1,$$

which means that $h(g(H^*)+C) - \beta h(\beta g(H^*)+C) > 0$. By equation (38) and the condition that $\frac{\Gamma(H^*)}{H^*} = 1$, inequality (44) shows that

$$\frac{H^0 + h(g(H^*)+C)}{\beta H^*} = \frac{1}{\beta g(H^*)} > \frac{H^0 + h(\beta g(H^*)+C)}{H^*}. \tag{45}$$

It follows that, if $H^0 < \min(h(m_1(H^*)+C), \tilde{H}^0)$,

$$\begin{aligned} \frac{h(2\beta g(H^*)+C_1') + h(\beta g(H^*)+C)}{H^*} + \frac{1}{\beta g(H^*)} &> \frac{h(2\beta g(H^*)+C_1') + 2h(\beta g(H^*)+C) + H^0}{H^*} \\ &= 1, \end{aligned}$$

where the last equality follows from equation (40). Hence, when $H^0 < \min(h(m_1(H^*)+C), \tilde{H}^0)$,

$$\frac{h(2\beta g(H^*)+C_1') + h(\beta g(H^*)+C)}{H^*} + \frac{1}{\beta g(H^*)} > 1,$$

which implies that

$$\frac{\beta g(H^*)-1}{\beta g(H^*)} < \frac{h(2\beta g(H^*)+C_1')+h(\beta g(H^*)+C)}{H^*}.$$

Therefore, the necessary and sufficient condition for $\hat{H}^* < H^*$ is satisfied.

We now inspect the changes in equilibrium prices caused by the cost reduction. When $\hat{H}^* > H^*$, assume for a contradiction that $\hat{g}(\hat{H}^*) \leq g(H^*)$. By equations (38) and (39), this implies that

$$\frac{h(2g(H^*)+C)+h(g(H^*)+C)}{H^*} \geq \frac{h(2\hat{g}(\hat{H}^*)+C_1')+h(\hat{g}(\hat{H}^*)+C)}{\hat{H}^*}. \tag{46}$$

By inspecting $\frac{\Gamma(H^*)}{H^*}=1$ and $\frac{\hat{\Gamma}(\hat{H}^*)}{\hat{H}^*}=1$, we obtain

$$\frac{h(g(H^*)+C)}{H^*} < \frac{h(\hat{g}(\hat{H}^*)+C)}{\hat{H}^*}. \tag{47}$$

Using inequalities (46) and (47), we see that

$$\frac{h(\hat{g}(\hat{H}^*)+C)}{h(g(H^*)+C)} > \frac{h(2\hat{g}(\hat{H}^*)+C_1')+h(\hat{g}(\hat{H}^*)+C)}{h(2g(H^*)+C)+h(g(H^*)+C)},$$

which implies that $\exp(-(\hat{g}(\hat{H}^*)-g(H^*))) > \exp\left(-(2(\hat{g}(\hat{H}^*)-g(H^*))+(C_1'-C))\right)$. This contradicts the assumption that $C_1'-C<0$ and $\hat{g}(\hat{H}^*) \leq g(H^*)$. Therefore, when $\hat{H}^* > H^*$, $\hat{g}(\hat{H}^*) > g(H^*)$. As we have shown that $\hat{g}(H) > g(H)$ for every $H > H^0$, where $\hat{g}(H)$ and $g(H)$ are strictly decreasing on (H^0,∞), it follows that when $\hat{H}^* \leq H^*$, $\hat{g}(\hat{H}^*) > g(H^*)$. Then, according to the pricing function $t_i(\Psi_i)=\Psi_i+C=P_i$ for any final product $i \in \{2,3\}$, the equilibrium prices of final products 2 and 3 always increase after a decrease in C_1.

All we need to do now is study the price change of final product 1. Rewriting equations (38) and (39) with $\frac{\Gamma(H^*)}{H^*}=\frac{\hat{\Gamma}(\hat{H}^*)}{\hat{H}^*}=1$ and $C_1 \in \{C, C_1'\}$, we obtain

$$\zeta(C_1,\mu) \equiv (\mu-2)h(\mu+C)+(\mu-1)H^0-h(2\mu+C_1)=0,$$

where $\mu = g(H^*)$ when $C_1 = C$, and $\mu = \hat{g}(\hat{H}^*)$ when $C_1 = C_1'$. Conversely, any solution in $\mu > 1$ of $\zeta(C_1,\mu)=0$ is an equilibrium with $C_1 \in \{C, C_1'\}$ and $H = H^0+h(2\mu+C_1)+2h(\mu+C)$. By Proposition 8, the solution is unique, and we denote it by $\mu^*(C_1)$. By Lemma 3, $\zeta(C_1,\mu)$ is continuous in $\mu > 0$ with $\zeta(C_1,1) < 0$ and $\lim_{\mu\to\infty}\zeta(C_1,\mu)=\infty$. Therefore, for any $C_1 \in \{C, C_1'\}$, $\zeta(C_1,\mu) < 0$ if and only if

$\mu < \mu^*(C_1)$, and $\zeta(C_1, \mu) > 0$ if and only if $\mu > \mu^*(C_1)$. This property implies that

$$\begin{aligned}\hat{t}_1(2\hat{g}(\hat{H}^*)) > t_1(2g(H^*)) &\iff 2\hat{g}(\hat{H}^*) + C_1' > 2g(H^*) + C \iff \hat{g}(\hat{H}^*) > g(H^*) + \frac{\Delta C_1}{2} \\ &\iff \zeta(C_1', g(H^*) + \frac{\Delta C_1}{2}) < 0.\end{aligned} \tag{48}$$

Note that since $\zeta(C, g(H^*)) = 0$,

$$h(2g(H^*) + \Delta C_1 + C_1') = h(2g(H^*) + C) = (g(H^*) - 2)h(g(H^*) + C) + (g(H^*) - 1)H^0,$$

which implies that

$$\zeta(C_1', g(H^*) + \frac{\Delta C_1}{2}) = h(g(H^*) + C)((g(H^*) - 2 + \frac{\Delta C_1}{2}) \exp{(\frac{-\Delta C_1}{2})} - (g(H^*) - 2)) + \frac{\Delta C_1}{2} H^0.$$

Therefore,

$$\begin{aligned}&\zeta(C_1', g(H^*) + \frac{\Delta C_1}{2}) < 0 \\ &\iff h(g(H^*) + C)((g(H^*) - 2 + \frac{\Delta C_1}{2}) \exp{(\frac{-\Delta C_1}{2})} - (g(H^*) - 2)) + \frac{\Delta C_1}{2} H^0 < 0 \\ &\iff H^0 < \bar{H}^0 = h(g(H^*) + C)(\frac{2(g(H^*) - 2)(1 - \exp(\frac{-\Delta C_1}{2}))}{\Delta C_1} - \exp(\frac{-\Delta C_1}{2})),\end{aligned}$$

which is the necessary and sufficient condition for the increase (or decrease) in the equilibrium price of final product 1.

We now prove that $\bar{H}^0 < \tilde{H}^0$, which is equivalent to showing

$$2(g(H^*) - 2)(1 - \exp(\frac{-\Delta C_1}{2})) - \Delta C_1 \exp{(\frac{-\Delta C_1}{2})} < 1 - \frac{1 + \Delta C_1}{\exp{(g(H^*)\Delta C_1)}}. \tag{49}$$

Inequality (49) clearly holds when $g(H^*) \leq 2$, as $\frac{1+\Delta C_1}{\exp{(g(H^*)\Delta C_1)}} < 1$ with $\Delta C_1 > 0$ and $g(H^*) > 1$ under Lemma 12. Let $Y_1(\Delta C_1) = 1 - \frac{1+\Delta C_1}{\exp{(g(H^*)\Delta C_1)}}$ and $Y_2(\Delta C_1) = 2(g(H^*) - 2)(1 - \exp(\frac{-\Delta C_1}{2})) - \Delta C_1 \exp{(\frac{-\Delta C_1}{2})}$. When $g(H^*) > 2$, inequality (43)

implies that $g(H^*) - 2 < \exp(-g(H^*))$, which shows that

$$\begin{aligned}
&Y_1(\Delta C_1) - Y_2(\Delta C_1) \\
&\quad \geq 1 - \frac{1+\Delta C_1}{\exp(g(H^*)\Delta C_1)} - 2\exp(-g(H^*)) + (\Delta C_1 + 2\exp(-g(H^*)))\exp(\frac{-\Delta C_1}{2}) \\
&\quad > 1 - \frac{1+\Delta C_1}{\exp(g(H^*)\Delta C_1)} - 2\exp(-g(H^*))(1 - \exp(\frac{-\Delta C_1}{2})) \\
&\quad > -\frac{1+\Delta C_1}{\exp(g(H^*)\Delta C_1)} + \exp(\frac{-\Delta C_1}{2}) \\
&\quad > -\frac{1+\Delta C_1}{\exp(2\Delta C_1)} + \exp(\frac{-\Delta C_1}{2}) > 0.
\end{aligned}$$

Therefore, inequality (49) holds for any $g(H^*) > 1$, which means that $\bar{H}^0 < \tilde{H}^0$.

Lastly, $\bar{H}^0 < h(m_1(H^*) + C)$ holds when $g(H^*) \leq 2$, as $\bar{H}^0 < 0$. When $g(H^*) > 2$,

$$\begin{aligned}
&\bar{H}^0 < h(m_1(H^*) + C) \\
&\iff (\frac{2(g(H^*)-2)(1-\exp(\frac{-\Delta C_1}{2}))}{\Delta C_1} - \exp(\frac{-\Delta C_1}{2})) < \exp(-g(H^*)).
\end{aligned}$$

Since

$$(\frac{2(g(H^*)-2)(1-\exp(\frac{-\Delta C_1}{2}))}{\Delta C_1} - \exp(\frac{-\Delta C_1}{2})) < \exp(-g(H^*))(\frac{2(1-\exp(\frac{-\Delta C_1}{2}))}{\Delta C_1}),$$

where $0 < (\frac{2(1-\exp(\frac{-\Delta C_1}{2}))}{\Delta C_1}) < 1$ for any $\Delta C_1 > 0$, we have $\bar{H}^0 < h(m_1(H^*) + C)$ when $g(H^*) > 2$. Therefore, $\bar{H}^0 < h(m_1(H^*) + C)$ for any $g(H^*) > 1$.

□

B.9 Proof of Proposition 8

Proof. Based on the proof of Theorem 1, the uniqueness of a pricing equilibrium under MNL demand can be proved by showing that the aggregate fitting-in function $\Gamma(H)$ has a unique fixed point on (H^0, ∞). This is equivalent to showing that $\frac{\Gamma(H)}{H} = 1$ has a unique solution in H on (H^0, ∞).

Under MNL demand, for every $i \in \mathcal{N}$, $\gamma_i = \frac{h_i'^2(P_i)}{h_i''(P_i)} = h_i(P_i)$, where $h_i(P_i) = \exp(\frac{a_i - P_i}{\lambda})$, $a_i \in \mathbb{R}$, and $\lambda > 0$. Since, by Lemma 10, $\Psi_i = \frac{P_i - C_i}{P_i}\eta_i(P_i) = \frac{P_i - C_i}{\lambda}$ for any $i \in \mathcal{N}$, we can derive the inverse function $t_i(\Psi_i) = \lambda\Psi_i + C_i$ for any $i \in \mathcal{N}$ on $(|M_i|, \infty)$. Let $S = \{S_1, S_2, ..., S_J\}$ be the set of separable components for any given network $(\mathcal{F}, \mathcal{N})$. Within the first separable component S_1, which has final products $\{1, 2, ..., K\} \subseteq \mathcal{N}$, by Lemma 12, final products' aggregate markups $(\Psi_i = m_i(H))_{i \in S_1}$ for every $H > 0$ solve the following system of equations derived from rewriting equa-

tion (14) for each firm in L_1:

$$\Psi_i = \sum_{f_k \in M_i} \frac{H}{H - \sum_{j \in G^k} \gamma_j(t_j(\Psi_j))}, \quad \forall i \in S_1. \tag{50}$$

To determine $m_i'(H)$ for every $i \in S_1$ and $H > 0$, we can apply implicit function theorem on equation (50) for all final products in S_1 to get the following equation:

$$\underbrace{\begin{bmatrix} 1 + (\sum_{f_l \in M_1} Z^l)h_1 & (\sum_{f_l \in (M_1 \cap M_2)} Z^l)h_2 & \cdots & (\sum_{f_l \in (M_1 \cap M_K)} Z^l)h_K \\ (\sum_{f_l \in (M_2 \cap M_1)} Z^l)h_1 & 1 + (\sum_{f_l \in M_2} Z^l)h_2 & \cdots & (\sum_{f_l \in (M_2 \cap M_K)} Z^l)h_K \\ \vdots & \vdots & \ddots & \vdots \\ (\sum_{f_l \in (M_K \cap M_1)} Z^l)h_1 & (\sum_{f_l \in (M_K \cap M_2)} Z^l)h_2 & \cdots & 1 + (\sum_{f_l \in M_K} Z^l)h_K \end{bmatrix}}_{\text{Denote this matrix by A}} \begin{bmatrix} m_1' \\ m_2' \\ \vdots \\ m_K' \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -(\sum_{f_l \in M_1} Z^l \frac{(\sum_{j \in G^l} h_j)}{H}) \\ -(\sum_{f_l \in M_2} Z^l \frac{(\sum_{j \in G^l} h_j)}{H}) \\ \vdots \\ -(\sum_{f_l \in M_K} Z^l \frac{(\sum_{j \in G^l} h_j)}{H}) \end{bmatrix}}_{\text{Denote this vector by } \mathbf{b}},$$

where $Z^k = \frac{H}{(H - \sum_{j \in G^k} h_j(t_j(\Psi_j)))^2} > 0$ for every $f_k \in L_1$ and $H > 0$. Then, we see that

$$\det(A) = \det(\underbrace{\begin{bmatrix} \frac{1}{h_1} + \sum_{f_l \in M_1} Z^l & \sum_{f_l \in (M_1 \cap M_2)} Z^l & \cdots & \sum_{f_l \in (M_1 \cap M_K)} Z^l \\ \sum_{f_l \in (M_2 \cap M_1)} Z^l & \frac{1}{h_2} + \sum_{f_l \in M_2} Z^l & \cdots & \sum_{f_l \in (M_2 \cap M_K)} Z^l \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{f_l \in (M_K \cap M_1)} Z^l & \sum_{f_l \in (M_K \cap M_2)} Z^l & \cdots & \frac{1}{h_K} + \sum_{f_l \in M_K} Z^l \end{bmatrix}}_{\text{Denote this matrix by } \bar{A}})(\prod_{j=1}^{K} h_j)$$

$$= \det(\bar{A})(\prod_{j=1}^{K} h_j).$$

For firm $f_l \in L_1$, let x^l be a vector with K entries, where the ith row is equal to 1 if firm f_l provides intermediates for final product $i \in S_1$, and 0 otherwise. We define matrix E^l for firm $f_l \in L_1$ to be

$$E^l = Z^l \begin{bmatrix} \chi_{G^l}(1) & \chi_{G^l}(1)\chi_{G^l}(2) & \cdots & \chi_{G^l}(1)\chi_{G^l}(K) \\ \chi_{G^l}(2)\chi_{G^l}(1) & \chi_{G^l}(2) & \cdots & \chi_{G^l}(2)\chi_{G^l}(K) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{G^l}(K)\chi_{G^l}(1) & \chi_{G^l}(K)\chi_{G^l}(2) & \cdots & \chi_{G^l}(K) \end{bmatrix} = Z^l(x^l {x^l}^T),$$

where $\chi_{G^k}(i)$ is an indicator function that is equal to 1 when $i \in G^k$. E^l identifies the final products that include firm f_l's intermediate products. It is obvious that matrix E^l is positive semi-definite. Then, we can decompose matrix $\bar{A}$ to be

$$\bar{A} = \text{Diag}(\frac{1}{h_1}, \frac{1}{h_2}, \cdots, \frac{1}{h_K}) + \sum_{f_l \in L_1} E^l.$$

By Lemma 12, $h_i(t_i(\Psi_i)) > 0$ for every $H > 0$. Therefore, matrix $\bar{A}$ is a summation of positive semi-definite matrix $\sum_{f_l \in L_1} E^l$ and a positive definite matrix $\text{Diag}(\frac{1}{h_1}, \frac{1}{h_2}, \cdots, \frac{1}{h_K})$, which means that $\bar{A}$ is positive definite and $\det(\bar{A}) > 0$ for every $\Psi_i \in (|M_i|, \infty)$ and $H > 0$. We now apply Cramer's rule to determine $m_1'(H)$ for every $H > 0$. We first calculate the determinant of matrix A_1, which is formed by replacing the first column of matrix A with vector $\mathbf{b}$:

$$\begin{aligned}
\det(A_1) &= \det\left(\begin{bmatrix} -(\sum_{f_l \in M_1} Z^l \frac{(\sum_{j \in G^l} h_j)}{H}) & (\sum_{f_l \in (M_1 \cap M_2)} Z^l) h_2 & \cdots & (\sum_{f_l \in (M_1 \cap M_K)} Z^l) h_K \\ -(\sum_{f_l \in M_2} Z^l \frac{(\sum_{j \in G^l} h_j)}{H}) & 1 + (\sum_{f_l \in M_2} Z^l) h_2 & \cdots & (\sum_{f_l \in (M_2 \cap M_K)} Z^l) h_K \\ \vdots & \vdots & \ddots & \vdots \\ -(\sum_{f_l \in M_K} Z^l \frac{(\sum_{j \in G^l} h_j)}{H}) & (\sum_{f_l \in (M_K \cap M_2)} Z^l) h_2 & \cdots & 1 + (\sum_{f_l \in M_K} Z^l) h_K \end{bmatrix}\right), \\
&= \det(\underbrace{\begin{bmatrix} (\sum_{f_l \in M_1} Z^l (\sum_{j \in G^l} h_j)) & (\sum_{f_l \in (M_1 \cap M_2)} Z^l) h_2 & \cdots & (\sum_{f_l \in (M_1 \cap M_K)} Z^l) h_K \\ (\sum_{f_l \in M_2} Z^l (\sum_{j \in G^l} h_j)) & 1 + (\sum_{f_l \in M_2} Z^l) h_2 & \cdots & (\sum_{f_l \in (M_2 \cap M_K)} Z^l) h_K \\ \vdots & \vdots & \ddots & \vdots \\ (\sum_{f_l \in M_K} Z^l (\sum_{j \in G^l} h_j)) & (\sum_{f_l \in (M_K \cap M_2)} Z^l) h_2 & \cdots & 1 + (\sum_{f_l \in M_K} Z^l) h_K \end{bmatrix}}_{\text{Subtract all other columns from the first column}})(-\frac{1}{H}), \\
&= \det\left(\begin{bmatrix} (\sum_{f_l \in M_1} Z^l) h_1 & (\sum_{f_l \in (M_1 \cap M_2)} Z^l) h_2 & \cdots & (\sum_{f_l \in (M_1 \cap M_K)} Z^l) h_K \\ (\sum_{f_l \in (M_2 \cap M_1)} Z^l) h_1 - 1 & 1 + (\sum_{f_l \in M_2} Z^l) h_2 & \cdots & (\sum_{f_l \in (M_2 \cap M_K)} Z^l) h_K \\ \vdots & \vdots & \ddots & \vdots \\ (\sum_{f_l \in (M_K \cap M_1)} Z^l) h_1 - 1 & (\sum_{f_l \in (M_K \cap M_2)} Z^l) h_2 & \cdots & 1 + (\sum_{f_l \in M_K} Z^l) h_K \end{bmatrix}\right)(-\frac{1}{H}), \\
&= \det\left(\begin{bmatrix} (\sum_{f_l \in M_1} Z^l) & (\sum_{f_l \in (M_1 \cap M_2)} Z^l) & \cdots & (\sum_{f_l \in (M_1 \cap M_K)} Z^l) \\ (\sum_{f_l \in (M_2 \cap M_1)} Z^l) - \frac{1}{h_1} & \frac{1}{h_2} + (\sum_{f_l \in M_2} Z^l) & \cdots & (\sum_{f_l \in (M_2 \cap M_K)} Z^l) \\ \vdots & \vdots & \ddots & \vdots \\ (\sum_{f_l \in (M_K \cap M_1)} Z^l) - \frac{1}{h_1} & (\sum_{f_l \in (M_K \cap M_2)} Z^l) & \cdots & \frac{1}{h_K} + (\sum_{f_l \in M_K} Z^l) \end{bmatrix}\right)(-\frac{1}{H} \prod_{j=1}^{K} h_j), \\
&= \det(\bar{A} + uv^T)(-\frac{1}{H} \prod_{j=1}^{K} h_j),
\end{aligned}$$

where

$$u = (-\frac{1}{h_1}) \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T, \quad \text{and} \quad v = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T.$$

We now apply the matrix determinant lemma to simplify $\det(A_1)$.

$$\det(A_1) = (1 + v^T(\bar{A})^{-1}u)\det(\bar{A})(-\frac{1}{H}\prod_{j=1}^{K} h_j) = (1 - \frac{1}{h_1}R_1(\bar{A}^{-1}))\det(\bar{A})(-\frac{1}{H}\prod_{j=1}^{K} h_j),$$

where the function $R_i(\bar{A}^{-1})$ denotes the sum of the entries in row i of matrix $\bar{A}^{-1}$. Therefore, we obtain

$$m_1'(H) = \frac{\det(A_1)}{\det(A)} = -\frac{1 - \frac{1}{h_1}R_1(\bar{A}^{-1})}{H}.$$

Similarly, for any final product $i \in S_1$,

$$m_i'(H) = \frac{\det(A_i)}{\det(A)} = -\frac{1 - \frac{1}{h_i}R_i(\bar{A}^{-1})}{H}.$$

Within the first separable component S_1,

$$\frac{d(\sum_{i\in S_1}\frac{h_i(t_i(m_i(H)))}{H})}{dH} = \frac{(\sum_{i\in S_1}(-h_i\frac{\partial m_i}{\partial H}H)) - \sum_{i\in S_1}h_i}{H^2} = -\frac{\sum_{i\in S_1}R_i(\bar{A}^{-1})}{H^2}.$$

The numerator in the equation above, $\sum_{i\in S_1}R_i(\bar{A}^{-1})$, is the summation of all entries in the matrix $\bar{A}^{-1}$. The fact that $\bar{A}$ is a positive definite matrix implies that $\bar{A}^{-1}$ is also a positive definite matrix, which has the property that $\sum_{i\in S_1}R_i(\bar{A}^{-1}) > 0$. As a result, we can determine that, for every $H > 0$,

$$\frac{d(\sum_{i\in S_1}\frac{h_i(t_i(m_i(H)))}{H})}{dH} < 0.$$

We then derive similar results for other separable components of the network, and conclude that

$$\frac{\Gamma(H)}{H} = \frac{\sum_{\alpha=1}^{J}\sum_{i\in S_\alpha}h_i(t_i(m_i(H))) + H^0}{H},$$

is strictly decreasing on (H^0, ∞) for any $H^0 > 0$. Therefore, by Lemma 13, the function $\Gamma(H)$ has a unique fixed point on (H^0, ∞), and the pricing equilibrium is unique under MNL demand. □

B.10 Proof of Proposition 9

Proof. By Theorem 1, the uniqueness of a pricing equilibrium under CES demand can be proved by showing that the aggregate fitting-in function $\Gamma(H)$ has a unique fixed point on (H^0, ∞). This is equivalent to showing $\frac{\Gamma(H)}{H} = 1$ has a unique solution $H^* \in (H^0, \infty)$. The method in the proof of Proposition 8 cannot be used with CES demand in

an arbitrary network, and we will use a different approach to show that $\frac{\Gamma(H)}{H}$ is strictly decreasing in H on (H^0, ∞).

Under CES demand, we know that by Lemma 3, $h_i(P_i) = \frac{\sigma}{\sigma-1}\gamma_i(P_i) = a_i P_i^{1-\sigma}$ for every $i \in \mathcal{N}$, where $a_i > 0$ and $\sigma > 1$. Assumption 2a also requires that $\lim_{P_i \to \infty} \eta_i(P_i) = \sigma > |M_i|$ for every $i \in \mathcal{N}$. Then, for every $H \in (H^0, \infty)$,

$$\frac{\Gamma(H)}{H} = \frac{\sigma}{\sigma-1}\frac{\sum_{i\in\mathcal{N}} \gamma_i(t_i(m_i(H)))}{H} + \frac{H^0}{H}.$$

We now show that $\frac{\Gamma(H)}{H}$ is strictly decreasing in $H \in (H^0, \infty)$ with CES demand under the conditions specified in Proposition 9. By Lemma 12, for every $H \geq H^0 > \underline{H}^0$, firms' η-markups $(\mu^k = g^k(H))_{f_k \in L_\alpha}$ in a separable component $S_\alpha \in S$ satisfy

$$\frac{\mu^k - 1}{\mu^k} = \frac{\sum_{i\in G^k} \gamma_i(t_i(\Psi_i))}{H}, \quad \forall f_k \in L_\alpha. \tag{51}$$

Then, Proposition 2 and Lemma 2 imply that, within any (**S**) or (**A1**) separable component $S_\alpha \in S$, there exists a firm $f_l \in L_\alpha$ such that $G^l = S_\alpha$ and $g^l(H)$ is strictly decreasing in H on (H^0, ∞). Let $\Gamma_\alpha(H) = \frac{\sigma}{\sigma-1}\sum_{i\in S_\alpha} \gamma_i(t_i(m_i(H)))$ for every $H \geq H^0$ and $S_\alpha \in S$. This shows that, by equation (51), in any (**S**) or (**A1**) separable component S_α,

$$\frac{\Gamma_\alpha(H)}{H} = \frac{\sigma}{\sigma-1}\frac{\sum_{i\in S_\alpha} \gamma_i(t_i(m_i(H)))}{H} = \frac{\sigma}{\sigma-1}\frac{g^l(H)-1}{g^l(H)},$$

which is strictly decreasing in H on (H^0, ∞).

We now study the case where $S_\alpha \in S$ is an (**A2**) network. It is obvious that $|L_\alpha| \geq 2$. We first show that when $|L_\alpha| = 2$, $\frac{d\frac{\Gamma_\alpha(H)}{H}}{dH} \equiv (\frac{\Gamma_\alpha}{H})' \leq 0$ for every $H > H^0$. Assume for a contradiction that $(\frac{\Gamma_\alpha}{H})' > 0$ at some $H = H' > H^0$. This means that there exists $i \in S_\alpha$ such that $(\frac{\gamma_i}{H})' \equiv \frac{d\frac{\gamma_i(t_i(m_i(H)))}{H}}{dH} > 0$ at $H = H'$, and we fix this final product i. By Lemmas 3, 10 and 12, for every $j \in S_\alpha$ and $H > H^0$, $|M_j| < m_j(H) < \sigma$, $\gamma_j(t_j(m_j(H))) > 0$, and

$$(\frac{\gamma_j}{H})' = \frac{\gamma_j(t_j(m_j(H)))(-\frac{\sigma-1}{\sigma-m_j(H)}Hm_j'(H) - 1)}{H^2}. \tag{52}$$

Hence, $m_i'(H') < -\frac{\sigma-m_i(H')}{(\sigma-1)H'} < 0$, which means that, by Lemma 12, there exists a firm $f_k \in M_i$ such that $(g^k(H'))' = \frac{dg^k(H)}{dH}\Big|_{H=H'} < 0$. Then, by equation (51) for firm f_k, $\sum_{j\in G^k}(\frac{\gamma_j}{H})' < 0$ at $H = H'$. Since $G^k \subseteq S_\alpha$ and $(\frac{\Gamma_\alpha}{H})' > 0$ at $H = H'$, there exists final product $q \in S_\alpha \setminus G^k$ such that $(\frac{\gamma_q}{H})' > 0$, and we fix this final product q. By the same argument for final product i, we know that $m_q'(H') < -\frac{\sigma-m_q(H')}{(\sigma-1)H'} < 0$, and there exists $f_l \in M_q$ such that $(g^l(H'))' < 0$. Also, since S_α is (**A2**) with $|L_\alpha| = 2$, $G^k \cap G^l \neq \varnothing$, $G^k \not\subseteq G^l$, and $G^l \not\subseteq G^k$. If, for every $j \in G^k \cap G^l$, $(\frac{\gamma_j}{H})' \geq 0$ at $H = H'$ or, for every

$y \in G^k \setminus G^l$, $(\frac{\gamma_y}{H})' \leq 0$ at $H = H'$, then, since $(g^k(H'))' < 0$ and $(g^l(H'))' < 0$ and by equation (51) for f_k and f_l, we can see that $\sum_{y \in G^k \cup G^l} (\frac{\gamma_y}{H})' < 0$ at $H = H'$, where $G^k \cup G^l = S_\alpha$. Hence, this contradicts the assumption that $(\frac{\Gamma_\alpha}{H})' > 0$ at $H = H'$, and there exists $j \in G^k \cap G^l$ such that $(\frac{\gamma_j}{H})' < 0$ at $H = H'$ and $y \in G^k \setminus G^l$ such that $(\frac{\gamma_y}{H})' > 0$ at $H = H'$. We fix such a final product j, and by equation (52), $m_j'(H') > -\frac{\sigma - m_j(H')}{(\sigma-1)H'}$. This implies that

$$m_j'(H') - m_q'(H') = (g^k(H'))' > \frac{m_j(H') - m_q(H')}{(\sigma-1)H'} = \frac{g^k(H')}{(\sigma-1)H'} > 0,$$

which contradicts the property that $(g^k(H'))' < 0$. Therefore, $(\frac{\Gamma_\alpha}{H})' \leq 0$ for every $H > H^0$.

Next, we show that when $|L_\alpha| = 3$, $(\frac{\Gamma_\alpha}{H})' \leq 0$ for every $H > H^0$. Assume for a contradiction that $(\frac{\Gamma_\alpha}{H})' > 0$ at some $H = H' > H^0$. We first show that there exist firms f_k and f_l in L_α such that $(g^k(H'))' < 0$, $(g^l(H'))' < 0$, $(\frac{\gamma_i}{H})' > 0$ at $H = H'$ for some $i \in G^k \setminus G^l$, denoted T^k, $(\frac{\gamma_q}{H})' > 0$ at $H = H'$ for some $q \in G^l \setminus G^k$, denoted T^l, and $(\frac{\gamma_j}{H})' < 0$ at $H = H'$ for some $j \in G^k \cap G^l$, denoted W^{kl}. Let P_α denote the set of final products $y \in S_\alpha$ with $(\frac{\gamma_y}{H})' > 0$ at $H = H'$, and let L_α^- denote the set of firms $f_k \in L_\alpha$ with $(g^k(H'))' < 0$. By the same argument as in the case with $|L_\alpha| = 2$, every $y \in P_\alpha$ is supplied by some firm in L_α^- and $|L_\alpha^-| \geq 2$. The same argument also identifies such firms when $P_\alpha \subseteq G^k \cup G^l$ for some $f_k, f_l \in L_\alpha^-$, which is the case when $|L_\alpha^-| = 2$. Hence, we only need to consider the case where $L_\alpha^- = L_\alpha$ and $P_\alpha \nsubseteq G^k \cup G^l$ for any two firms $f_k, f_l \in L_\alpha$. Then, for any two firms $f_k, f_l \in L_\alpha$, with f_m denoting the remaining firm, a final product in $P_\alpha \setminus (G^l \cup G^m)$ belongs to $G^k \setminus G^l$, so $T^k \neq \varnothing$ and, similarly, $T^l \neq \varnothing$. Moreover, since $(g^k(H'))' < 0$ for every $f_k \in L_\alpha$, equation (51) implies that $\sum_{f_k \in L_\alpha} \sum_{z \in G^k} (\frac{\gamma_z}{H})' = \sum_{z \in S_\alpha} |M_z| (\frac{\gamma_z}{H})' < 0$ at $H = H'$, whereas $(\frac{\Gamma_\alpha}{H})' = \frac{\sigma}{\sigma-1} \sum_{z \in S_\alpha} (\frac{\gamma_z}{H})' > 0$ at $H = H'$. Therefore, $\sum_{z \in S_\alpha} (|M_z| - 1)(\frac{\gamma_z}{H})' < 0$ at $H = H'$, so there exists $j \in S_\alpha$ with $|M_j| \geq 2$ and $(\frac{\gamma_j}{H})' < 0$ at $H = H'$, and any two firms $f_k, f_l \in M_j$ satisfy $j \in W^{kl}$. In the following, i, q, and j denote arbitrary elements of T^k, T^l, and W^{kl}.

We now study the intermediates offered by the third firm f_m and argue that $T^k \cup T^l \subseteq G^m$. Assume for a contradiction that $i \in T^k$ and $i \notin G^m$. Since $(\frac{\gamma_i}{H})' > 0$ at $H = H'$ and $(\frac{\gamma_j}{H})' < 0$ at $H = H'$, this implies that, by equation (52),

$$m_i'(H') < -\frac{\sigma - m_i(H')}{(\sigma-1)H'}, \quad m_j'(H') > -\frac{\sigma - m_j(H')}{(\sigma-1)H'}, \quad \text{and}$$
$$m_j'(H') - m_i'(H') > \frac{m_j(H') - m_i(H')}{(\sigma-1)H'}.$$

Then, we must have $j \in G^m$ and

$$m_j'(H') - m_i'(H') = (g^m(H'))' + (g^l(H'))' > \frac{g^m(H') + g^l(H')}{(\sigma-1)H'} > 0.$$

Since $(g^l(H'))' < 0$, $(g^m(H'))' > 0$. By equation (51), this implies that there exists $y \in G^m$ such that $(\frac{\gamma_y}{H})' > 0$ at $H = H'$. Since $(g^m(H'))' > 0$, $|L_\alpha| = 3$, and $m_y'(H') < -\frac{\sigma - m_y(H')}{(\sigma-1)H'} < 0$ by equation (52), we see that $y \notin S_\alpha \setminus (G^k \cup G^l)$ and $y \in G^k \cup G^l$. It follows that

$$m_j'(H') - m_y'(H') > \frac{m_j(H') - m_y(H')}{(\sigma-1)H'}. \tag{53}$$

Note that since $j, y \in G^m$, $j \in G^k \cap G^l$, and $y \in G^k \cup G^l$, inequality (53) cannot hold. Therefore, $T^k \cup T^l \subseteq G^m$. Similarly, if $z \in G^k \cap G^l$ and $(\frac{\gamma_z}{H})' > 0$ at $H = H'$, then $z \in G^m$.

Let T^{kl} denote final products $z \in G^k \cup G^l$ such that $(\frac{\gamma_z}{H})' > 0$ at $H = H'$. We now know that $T^{kl} \subseteq G^m$. If $(g^m(H))' = (g^m(H))^2(\sum_{y\in G^m}(\frac{\gamma_y}{H})') \leq 0$ at $H = H'$ by equation (51), then since, $T^{kl} \subseteq G^m$, and any $y \in (G^k \cup G^l) \setminus G^m$ satisfies $y \notin T^{kl}$ with $(\frac{\gamma_y}{H})' \leq 0$ at $H = H'$,

$$(\frac{\Gamma_\alpha}{H})' = \frac{\sigma}{\sigma-1}(\sum_{y\in G^m}(\frac{\gamma_y}{H})' + \sum_{y\in(G^k\cup G^l)\setminus G^m}(\frac{\gamma_y}{H})') \leq 0 \quad \text{at } H = H'.$$

This contradicts the assumption that $(\frac{\Gamma_\alpha}{H})' > 0$ at $H = H'$. Hence, at $H = H'$, $(g^m(H))' = (g^m(H))^2(\sum_{y\in G^m}(\frac{\gamma_y}{H})') > 0$. Since $m_j'(H') - m_y'(H') > \frac{m_j(H') - m_y(H')}{(\sigma-1)H'}$ for $j \in W^{kl}$ and $y \in T^k$, we see that $j \notin G^m$, which implies that $(g^l(H'))' - (g^m(H'))' > \frac{g^l(H') - g^m(H')}{(\sigma-1)H'}$. Since $(g^l(H'))' - (g^m(H'))' < 0$, it follows that $g^l(H') < g^m(H')$. For any $q \in T^l$, by equation (52), $m_q'(H') = (g^l(H'))' + (g^m(H'))' < -\frac{\sigma - m_q(H')}{(\sigma-1)H'} < 0$. Then, by equation (51) for f_l and f_m, $(g^l(H))' = (g^l(H))^2(\sum_{y\in G^l}(\frac{\gamma_y}{H})') < 0$ at $H = H'$ and $(g^m(H))' = (g^m(H))^2(\sum_{y\in G^m}(\frac{\gamma_y}{H})') > 0$ at $H = H'$. Therefore, at $H = H'$ with $1 < g^l(H') < g^m(H')$, $(\sum_{y\in G^l}(\frac{\gamma_y}{H})') + (\sum_{y\in G^m}(\frac{\gamma_y}{H})') < 0$. Note that since $T^{kl} \subseteq G^m$, any $z \in (G^l \cap G^m) \setminus G^k$ must have $(\frac{\gamma_z}{H})' > 0$ at $H = H'$ by equation (52). Then, since $(g^k(H'))' < 0$, $g^k(H') > 1$, and $(\frac{\gamma_z}{H})' > 0$ for any $z \in (G^l \cap G^m) \setminus G^k$ at $H = H'$, equation (52) also implies that any $y \in (G^l \cap G^m \cap G^k)$ must have $(\frac{\gamma_y}{H})' > 0$. Therefore, we see that, at $H = H'$,

$$\begin{aligned}
(\frac{\Gamma_\alpha}{H})' &= \frac{\sigma}{\sigma-1}(\sum_{y\in G^k\cup G^l\cup G^m}(\frac{\gamma_y}{H})') \\
&= \frac{\sigma}{\sigma-1}(\sum_{y\in G^l}(\frac{\gamma_y}{H})' + \sum_{y\in G^m}(\frac{\gamma_y}{H})' - \sum_{y\in G^l\cap G^m}(\frac{\gamma_y}{H})' + \sum_{y\in G^k\setminus(G^l\cup G^m)}(\frac{\gamma_y}{H})') \\
&< 0,
\end{aligned}$$

where $\sum_{y \in G^k \setminus (G^l \cup G^m)} (\frac{\gamma_y}{H})' \leq 0$ because $T^{kl} \subseteq G^m$. This contradicts $(\frac{\Gamma_\alpha}{H})' > 0$ at $H = H'$. Hence, $(\frac{\Gamma_\alpha}{H})' \leq 0$ for every $H > H^0$.

Finally, suppose that S_α is (**A2**), $|S_\alpha| = 3$, and $G^k \neq G^l$ for any $f_k, f_l \in L_\alpha$, and assume for a contradiction that $(\frac{\Gamma_\alpha}{H})' > 0$ at some $H = H' > H^0$. By the same argument as in the case with $|L_\alpha| = 2$, there exist $f_k, f_l \in L_\alpha$ with $(g^k)' < 0$ and $(g^l)' < 0$ that supply two different final products $y, z \in S_\alpha$ with $(\frac{\gamma_y}{H})' > 0$ and $(\frac{\gamma_z}{H})' > 0$ at $H = H'$. Since $(\frac{\Gamma_\alpha}{H})' > 0$ at $H = H'$, equation (51) implies that each of f_k and f_l supplies exactly one other final product $w \in S_\alpha$ with $(\frac{\gamma_w}{H})' < 0$. Hence, we can assume without loss of generality that $S_\alpha = \{1,2,3\}$, $G^k = \{1,2\}$, $G^l = \{2,3\}$, $(g^k)' < 0$, $(g^l)' < 0$, $(\frac{\gamma_1}{H})' > 0$, $(\frac{\gamma_2}{H})' < 0$, and $(\frac{\gamma_3}{H})' > 0$ at $H = H'$. Let f_m, f_{x1}, f_{x2}, f_{x3}, and f_{x4} denote the firms with $G^m = \{1,3\}$, $G^{x1} = \{1\}$, $G^{x2} = \{2\}$, $G^{x3} = \{3\}$, and $G^{x4} = S_\alpha$ when they are in L_α, and set $g^f \equiv 0$ for those that are not. By equation (51), $(g^m)' > 0$, $(g^{x1})' > 0$, $(g^{x3})' > 0$, and $(g^{x4})' > 0$, and $(g^{x2})' < 0$ at $H = H'$ when these firms are in L_α. By equation (52) for final products 1 and 2, $m_2' - m_1' > \frac{m_2 - m_1}{(\sigma-1)H'}$, where $m_2' - m_1' = (g^l)' + (g^{x2})' - (g^m)' - (g^{x1})' < 0$. Then, we know that $m_2 < m_1$, which means that

$$g^l + g^{x2} < g^m + g^{x1}, \quad \text{and similarly,} \quad g^k + g^{x2} < g^m + g^{x3}. \tag{54}$$

Equation (52) for final products 1, 2, and 3 also gives

$$\begin{aligned} m_1' &= (g^{x1})' + (g^k)' + (g^m)' + (g^{x4})' < -\frac{\sigma - m_1}{(\sigma-1)H'} < 0, \\ m_2' &= (g^k)' + (g^l)' + (g^{x2})' + (g^{x4})' > -\frac{\sigma - m_2}{(\sigma-1)H'}, \\ m_3' &= (g^{x3})' + (g^l)' + (g^m)' + (g^{x4})' < -\frac{\sigma - m_3}{(\sigma-1)H'} < 0. \end{aligned} \tag{55}$$

We distinguish three cases.

Case 1: $f_m \notin L_\alpha$. Equation (54) shows that $g^l < g^{x1}$ and $g^k < g^{x3}$. Then, by equation (51) at $H = H'$, $\gamma_2 + \gamma_3 < \gamma_1$ and $\gamma_1 + \gamma_2 < \gamma_3$, contradicting $\gamma_2 > 0$.

Case 2: $f_m \in L_\alpha$ and $g^m > g^l$ (the case $g^m > g^k$ is symmetric). The third inequality in (55) gives $(g^m)' < -(g^l)'$, so $\frac{(g^m)'}{(g^m)^2} < \frac{-(g^l)'}{(g^l)^2}$. By equation (51), this implies that $\sum_{z \in G^m} (\frac{\gamma_z}{H})' + \sum_{z \in G^l} (\frac{\gamma_z}{H})' < 0$ at $H = H'$. Since $G^m \cup G^l = S_\alpha$ and $G^m \cap G^l = \{3\}$, this shows that $\frac{\sigma-1}{\sigma}(\frac{\Gamma_\alpha}{H})' + (\frac{\gamma_3}{H})' < 0$, contradicting $(\frac{\gamma_3}{H})' > 0$.

Case 3: $f_m \in L_\alpha$ and $g^m \leq \min(g^k, g^l)$. By equation (54), $g^{x1} > g^l - g^m + g^{x2} \geq g^{x2}$ and similarly, $g^{x3} > g^{x2}$. Hence, $f_{x1}, f_{x3} \in L_\alpha$, and $f_{x2} \notin L_\alpha$, because $g^{x1} > g^{x2}$ with $f_{x2} \in L_\alpha$ would lead to $\gamma_1 > \gamma_2$ and thus $g^m > g^l$ by equation (51). Therefore, any firm in L_α is in M_1 or M_3, and $|M_1| = |M_3| = q$ with $q \in \{3,4\}$ (the existence of f_{x4} affects q, but it does not affect the remaining analysis). Let $\bar{g} = \max(g^k, g^l)$. By equation (51) for f_k, f_l, and f_m,

$$\frac{2(\sigma-1)}{\sigma}\Big(\frac{\Gamma_\alpha}{H}\Big)' = \frac{(g^k)'}{(g^k)^2} + \frac{(g^l)'}{(g^l)^2} + \frac{(g^m)'}{(g^m)^2} > 0,$$

which gives $(g^m)' > (g^m)^2(\frac{-(g^k)'}{(g^k)^2} + \frac{-(g^l)'}{(g^l)^2}) \geq (\frac{g^m}{\bar{g}})^2(-(g^k)' - (g^l)')$. Combining the first and third inequalities in (55) and using $(g^{x1})', (g^{x3})' > 0$ gives $2(g^m)' < -(g^k)' - (g^l)' - 2(g^{x4})' - \frac{2\sigma - m_1 - m_3}{(\sigma-1)H'}$. Therefore,

$$(1 - 2(\frac{g^m}{\bar{g}})^2)(-(g^k)' - (g^l)') > 2(g^{x4})' + \frac{2\sigma - m_1 - m_3}{(\sigma-1)H'} > 0,$$

which implies that $1 - \frac{2(g^m)^2}{\bar{g}^2} > 0$ as $(g^k)' < 0$ and $(g^l)' < 0$. Since $-(g^k)' - (g^l)' < (g^{x4})' + \frac{\sigma - m_2}{(\sigma-1)H'}$ by the second inequality in (55), it follows that $(1 - 2(\frac{g^m}{\bar{g}})^2)((g^{x4})' + \frac{\sigma - m_2}{(\sigma-1)H'}) > 2(g^{x4})' + \frac{2\sigma - m_1 - m_3}{(\sigma-1)H'}$, and therefore, as $0 < 1 - 2(\frac{g^m}{\bar{g}})^2 < 2$ and $(g^{x4})' \geq 0$,

$$m_1 + m_3 - m_2 - \sigma > 2(\frac{g^m}{\bar{g}})^2(\sigma - m_2). \tag{56}$$

As shown in the proof of Lemma 12, Assumption 2a and the definition of $\underline{H}^0$ imply that $g^f(H') < \underline{\mu}^f = \min_{i \in G^f} \frac{\bar{\mu}_i}{|M_i|} = \frac{\sigma}{q}$ for every $f \in L_\alpha$ at $H = H'$. Since $m_2 = g^k + g^l + g^{x4}$ and $|M_2| = q - 1$, $\sigma - m_2 > \frac{\sigma}{q} > \bar{g}$. Since $m_1 + m_3 - m_2 = g^{x1} + g^{x3} + 2g^m + g^{x4}$, with $g^{x1}, g^{x3} < g^m$ by equation (51) and $\sigma > (\max(\bar{g}, g^{x4}))q \geq 3\bar{g} + g^{x4}$, we have $m_1 + m_3 - m_2 - \sigma < 4g^m - 3\bar{g}$. Then, by inequality (56) and $\sigma - m_2 > \bar{g}$, $4g^m - 3\bar{g} > \frac{2(g^m)^2}{\bar{g}}$, which is equivalent to $3\bar{g}^2 - 4g^m\bar{g} + 2(g^m)^2 < 0$. This cannot be satisfied because $3\bar{g}^2 - 4g^m\bar{g} + 2(g^m)^2 = 3(\bar{g} - \frac{2}{3}g^m)^2 + \frac{2}{3}(g^m)^2 > 0$. This contradiction completes the proof of (c), and hence $(\frac{\Gamma_\alpha}{H})' \leq 0$ for every $H > H^0$.

In summary, if every $S_\alpha \in S$ satisfies the conditions in Proposition 9, then $(\frac{\Gamma_\alpha}{H})' \leq 0$ on (H^0, ∞), which means that, at every $H > H^0$,

$$\frac{d\frac{\Gamma(H)}{H}}{dH} = \frac{d(\frac{H^0}{H} + \sum_{S_\alpha \in S} \frac{\Gamma_\alpha(H)}{H})}{dH} = -\frac{H^0}{H^2} + \sum_{S_\alpha \in S} (\frac{\Gamma_\alpha}{H})' < 0.$$

Therefore, $\frac{\Gamma(H)}{H}$ is strictly decreasing on (H^0, ∞), and by Lemma 13, $\frac{\Gamma(H)}{H} = 1$ has a unique solution $H^* \in (H^0, \infty)$. □

B.11 Example 2: Multiple Pricing Equilibria under CES Demand

Consider a market with one separable component, where $\mathcal{N} = \{1, 2, 3\}$ and $\mathcal{F} = \{f_1, f_2, f_3, f_4, f_5, f_6\}$. Also, $G^1 = \{1, 2\}$, $G^2 = \{2, 3\}$, $G^3 = \{1\}$, $G^4 = \{1\}$, $G^5 = \{3\}$, and $G^6 = \{3\}$. The demand for any final product $i \in \mathcal{N}$ follows the CES demand specification with $h_i(P_i) = a_i P_i^{1-\sigma}$, where $\sigma = 4.95$ and $a_1 = a_2 = a_3 = 1$. Moreover,

$H^0 = 0.00003$, $C_1 = 0.01$, $C_2 = 0.1$, and $C_3 = 0.01$. According to Lemma 11, firm-level costs affect the pricing equilibria only through the total costs of final products, so we do not specify firm-level costs here, and we focus on the equilibrium prices of the final products (P_i^*), firms' equilibrium η-markups (μ^{k*}), and the equilibrium consumer surplus ($\log(H^*)$) in this example. $\mu^{k*} - 1$ for any $f_k \in \mathcal{F}$ also gives firm f_k's equilibrium profit.

Equation (13) in Lemma 11 has three solutions, corresponding to three pricing equilibria, as shown in the table below. In all three pricing equilibria, the choice probabilities of the outside option are below 0.61%.

Table 1: Three pricing equilibria in this market.

$\log(H^*)$	$P_1^* = P_3^*$	P_2^*	$\mu^{1*} = \mu^{2*}$	μ^{x*} $(x \in \{3,4,5,6\})$
−5.299131	5.387957	4.612773	2.421345	1.259734
−2.323932	2.492937	2.209501	2.362984	1.283580
3.817280	0.489463	0.534694	2.012118	1.418375

Note that by setting $H = H^* = 45.480354$, which corresponds to $\log(H^*) = 3.817280$, firm f_1's equation (14) has no solution with $\Psi_1 < \sigma$ when $\mu^2 = 1$ and $\mu^3 = \mu^4 = 1.4$, as its right-hand side is strictly greater than its left-hand side for every $\mu^1 \in (1, \sigma - \mu^3 - \mu^4)$. Its solution with the extension $t_1(\Psi_1) = \infty$ for $\Psi_1 \geq \sigma$ by Lemma 10 is $\mu^1 = 2.723118$, with $\Psi_1 = 5.523118 > \sigma$ and $\Psi_2 = 3.723118 < \sigma$. Therefore, firm f_1 chooses an infinite price for final product 1 in response to $H = 45.480354 > H^0$ and finite η-markups and prices chosen by other firms.