

An Extreme-Points Approach to Multidimensional Delegation: 1,2,3, ∞^*

Patrick Lahr[†]

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Abstract

A principal designs a potentially random decision rule selecting between k outcomes. An agent has private information about the state of the world, determining both von Neumann-Morgenstern utility functions. I place no further restrictions on preferences. The design problem reduces to selecting an optimal convex menu of lotteries from which the agent chooses his preferred one. I characterize the extreme points of the set of such menus as maximal and indecomposable subsets of the unit simplex. In particular, for three outcomes there is always an optimal mechanism with at most a range of three, yet extreme points lie dense in the set of maximal menus for four or more outcomes. My results are related to previously observed phenomena in the multi-object monopolistic seller problem. My analysis rests on the literature on indecomposable convex bodies started by Gale [1954]. Applications include job design, the allocation of public housing, and hiring.

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[†]University of Bonn, patricklahr@uni-bonn.de

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1 Introduction

The allocation of decision rights is a fundamental question for organizational design. An essential consideration is the trade-off between the efficiency gains from delegating a decision to a well-informed agent and the agency costs created by conflicts of interest with said agents. I study a simple model of such delegation in which a principal (she) has to make a potentially random decision between a finite set of outcomes, affecting both her and an agent (he). The agent is privately informed about the state of the world, which determines both players' preferences. The agent's preferences are drawn from the entire domain of von Neumann-Morgenstern (vNM) preferences over lotteries. The taxation principle implies that the principal is effectively limited to delegating her decision to the agent while restricting the menu of feasible random choices.

Examples of this setting include, delegation of hiring decisions between the central office and a local branch in a company, allocation of housing by the government and a manager designing a job by specifying permissible time allocations on tasks.

This model also has a systematic role within mechanism design theory. It subsumes many well-known one-agent mechanism design problems. To illustrate this, consider the case with 2^{k+1} outcomes. Each outcome can represent a potential allocation of k goods and a sufficiently large lump sum transfer, the probability of its allocation taking the role of traditional transfers. My model then becomes an instance of the multi-good monopolistic seller problem without requiring the agent's utility function to satisfy, e.g., additive separability or free disposal. I will frequently refer to this connection. Robust observations like that there is no distortion at the top, the optimality of post-it price mechanisms in the one good case or the complexities implied by the possibility of bundling are all special cases of more general corresponding properties in my model.

I characterize the extreme points of the set of direct incentive-compatible mechanisms in this setting. The role of extreme points rests on two central results: First, by Bauer's maximum principle, the set of extreme points is a

sufficient candidate set whenever one maximizes a linear or convex functional. In my model, utilities are linear since the principal and agent have vNM and hence linear preferences. Second, Choquet’s theorem implies that properties preserved under convex combinations generalize from the extreme points to the entire set. They are hence also a sufficient test set if one tries to establish such a property for all mechanisms.

To these fundamental properties, I want to add a simple observation relating to the more recent discussion around simple mechanisms started by Hart and Nisan [2019]: Extreme points are menu size efficient, i.e., for a given limit on the number of different allocations an agent might receive, Bauer’s maximum principle also generalizes to the problem, including this side constraint. Even when it is known that extreme points can generally be arbitrarily complex, a characterization of them can hence aid in trading-off complexity and efficiency.

Results. I show that the extreme points of the set of incentive-compatible mechanisms in my model correspond to maximal and indecomposable convex bodies in the simplex of probability distributions over k outcomes. Before I explain the content of this characterization, I will point out three immediate consequences:

First, extreme points are either constant and deterministic or give the agent a veto. Consequently, any extreme points with range two consist of lotteries with disjoint supports.

Second, for three outcomes, non-constant extreme points give the agent a veto and have a range of two or three.

Lastly, for more than three outcomes, the set of extreme points lies dense in the set of mechanisms granting a veto. However, those mechanisms that are extreme points with finite but arbitrary ranges are themselves a dense set of mechanisms granting a veto. I will hence focus attention on these. They correspond to polytopes in my characterization and are generally much better understood, which aids in range-restricted optimization problems.

A convex set is said to be indecomposable if it has no representation as a Minkovski sum of two convex sets, both not homothetic¹ to the sum.

¹Two convex set $K, L \subset \mathbb{R}^d$ are homothetic if $K = \alpha L + b$, with $\alpha \in \mathbb{R}^+$ and $b \in \mathbb{R}^d$

A Minkovski sum is the set of all pairwise sums of its summands. Both indecomposable sets and Minkovski sums are widely studied objects in convex geometry. Grünbaum [2003] is an excellent reference. It is well understood and easy to check algebraically whether a two-dimensional convex body or any polytope is indecomposable. Further, I call a set maximal in the simplex if it is either a vertex of the simplex or touches all facets².

My characterization follows from separate characterizations. Both are of independent interest. Note that the revelation and taxation principles apply in this setting. Hence any implementable decision rule can be implemented both by a direct mechanism and indirectly by letting the agent decide from a menu. Since the agent has linear preferences, I can restrict attention to menus that are convex bodies, i.e., convex, compact, and non-empty subsets of probability distributions over k outcomes.³ Given a direct mechanism, one can easily describe the associated menu and vice versa. I strengthen this equivalence of the two principles by demonstrating the existence of an isomorphism between direct mechanisms and menus that preserves convex combinations. In particular, the set of menus has a convex structure, where convex combinations of menus refer to the Minkovski summation as defined above. Note that I here and below refer to a convex set of convex sets. Except in specifically noted circumstances, mentions of extreme points will refer to the objects within this meta-structure. This directly yields theorem 1: A direct mechanism is an extreme point if and only if its associated menu is an extreme point.

For the second equivalence, I study the above menus as geometric objects. These are convex bodies within the unit simplex. A convex body can be represented as a convex combination of other convex bodies in the simplex homothetic to the first if and only if it is non-maximal. Hence, maximality in the simplex is necessary for convex bodies to be an extreme point.

In contrast, if a convex body in the simplex is indecomposable, any representation as a convex combination must, in turn, imply that the parts are

²The facets of the simplex represent the subsets of all lotteries for which a given outcome has probability zero.

³An agent can always select an option from the convex hull by randomizing over reports, yet will never have the incentive to do so.

homothetic to the original. Hence maximality and indecomposability combined are sufficient for convex bodies to be an extreme point. The reverse is also true. This yields Theorem 2: A convex body of probability distributions over a finite set of outcomes is an extreme point of the set of all such bodies if and only if it is maximal and indecomposable. Maximality has a simple economic interpretation: A maximal menu either leaves no choice at all, i.e., always implements the same outcome, or it grants a veto, i.e., the agent can make sure that one outcome never realizes. Jointly with theorem 1, it implies my main characterization: A direct non-constant mechanism is an extreme point if it grants a veto and has an indecomposable menu.

The above mentioned direct consequences are then all direct consequences of the literature on indecomposable sets.

Related Literature. Holmstrom [1984] has initiated a vast field of research on delegation. Most of this literature has focused on a one-dimensional action and type space and parameterized, mostly quadratic loss utility functions, e.g., Dessein [2002], Alonso and Matouschek [2008], Amador and Bagwell [2013], and Kolotilin and Zapechelnyuk [2019].

I deviate from this main strand in two ways: The set of alternatives consists of lotteries over k outcomes, and arbitrary vNM preferences are permissible. Both the space of alternatives and preferences are, therefore, multi-dimensional and compact, and the utility functions of both players are linear.

A smaller number of publications also consider multi-dimensional types or action spaces. These include Bendor and Meirowitz [2004], Koessler and Martimort [2012], Frankel [2016] and Kleiner [2022].

Lastly, papers that study delegation over a finite set of outcomes include Che et al. [2013], Nocke and Whinston [2013], and Armstrong and Vickers [2010]. In these models, the agent selects one of the alternatives for the principal. Upon selection, the principal receives a signal on its quality and can accept or reject the agent's recommendation. The anticipation of this signal acts as a screening device. My model shows that commitment to enacting the agent's recommendation contingent on a random event can be effective, even if this event is completely independent of the relevant state of

the world.

The closest paper to the present is, however, on auctions. Manelli and Vincent [2007] characterize the extreme points of the multi-good monopolistic seller problem. Given the differences between the models, also the characterizations differ substantially. I will discuss the relationship in the main below. Kleiner et al. [2021] characterize extreme points of the monotone functions that fulfill a majorization constraint and apply their characterization to several one-dimensional economic design problems. Both papers apply a different set of methods but share the approach to directly studying the convex structure of the set of mechanisms. In contrast, I apply an indirect approach via the convex structure of feasible menus. The approaches are, of course, deeply related, yet the indirect approach allows me to apply elegant results from convex geometry, which fits my model exactly.

More generally, my model is deeply connected to models of multi-dimensional mechanism design and, in particular, the multi-object monopolistic seller problem, studied in, e.g., Rochet and Choné [1998], Jehiel et al. [2007], Daskalakis et al. [2015], Hart and Reny [2015] and Haghpanah and Hartline [2021].

Recent work in this literature started by Hart and Nisan [2019] has focused on simple mechanisms in the sense that they have a small range of outcomes. However, there is a tight connection to the study of extreme points since extreme points are menu size efficient, i.e., when restricting to mechanisms with menus lower than a given size, extreme points will retain their status as a sufficient candidate set. Hart and Nisan [2017] demonstrate that for two goods and arbitrary correlation structures, finite mechanisms may not secure any positive fraction of optimal revenue. Since their model is a special case of mine, this result directly translates to my model if $k \geq 8$, yet I conjecture it to be true for $k \geq 4$. Finally, Babaioff et al. [2017] study how fast optimal revenue can be approximated by finite menu mechanisms when valuations for different goods are understood to be independent.

2 Model

2.1 Notation

$Conv(\cdot)$ and $\overline{Conv}(\cdot)$ denote the convex hull and closed convex hull of a set, respectively. $Ext(\cdot)$ denotes the set of extreme points. Scalar multiplication and addition of sets of real vectors refer to the following operations:

$$\lambda M = \{\lambda m | m \in M\}$$

and

$$M + M' = \{m + m' | m \in M, m' \in M'\}.$$

The ”+” here is the standard definition of the Minkovski or vector sum. $M \sim M'$ will denote that M and M' are homothetic.

2.2 Setting

A principal (she) selects an alternative affecting herself and an agent (he). The set of alternatives A is the set of probability distributions over some finite set of outcomes $A = \Delta\{1, \dots, k\}$. Hence an element $a \in A$ is of the form $a = (a_1, \dots, a_k)$, where $a_i \geq 0$ for $i = 1, 2, \dots, k$ and $\sum_{i=1}^k a_i = 1$. The agent is privately informed about his type $\theta = (\theta_1, \dots, \theta_k) \in \Theta = \{[0, 1]^k : \max_i \theta_i = 1 \text{ and } \min_i \theta_i = 0\}$ which represents his (normalized) Bernoulli utilities over outcomes. It is drawn from some prior μ on Θ . Hence his utility functions U reads.

$$U(a, \theta) = a \cdot \theta$$

I also will assume throughout that the principal is a von Neumann-Morgenstern expected utility maximizer, i.e., given a type θ , her preferences over lotteries are characterized by her Bernoulli utilities over outcomes $v(\theta) = (v_1(\theta), \dots, v_k(\theta))$. The principal's ex-ante utility function V is then given by:

$$V(a) = \mathbb{E}_\mu [a \cdot v(\theta)]$$

2.3 Mechanisms

The principal commits to a mechanism, and I assume the agent plays a best response.

A mechanism is given by a message space S and a choice functions $f : S \rightarrow A$ that specifies an alternative for each message sent by the agent.

Due to the revelation principle, it is without loss to restrict attention to direct mechanisms, i.e., mechanisms with $S = \Theta$ and where agents have incentives to report their true type. Formally, a mechanism satisfies incentive compatibility if

$$U(f(\theta), \theta) \geq U(f(\theta'), \theta) \text{ for each } \theta, \theta' \in \Theta. \quad (\text{IC})$$

When the agent is indifferent between the alternative assigned to his type and another, I assume that the principal can select which best response is played by the agent.⁴

Definition 1. *A mechanism with choice rule f satisfies principal preferred tie-breaking if the following conditions are satisfied.*

- (i) *For any $\theta, \theta' \in \Theta$ s.t. $U(f(\theta), \theta) = U(f(\theta'), \theta)$ implies $V(f(\theta), \theta) \geq V(f(\theta'), \theta)$.*
- (ii) *For any $\theta, \theta' \in \Theta$ s.t. $U(f(\theta), \theta) = U(f(\theta'), \theta)$ and $V(f(\theta), \theta) = V(f(\theta'), \theta)$, implies $f(\theta) \geq_{lex} f(\theta')$, where " \geq_{lex} " refers to the lexicographical order.*

Henceforth, when considering direct mechanisms, I will restrict attention to those that satisfy incentive compatibility and principal preferred tie-breaking. I will denote the set of choice rules implemented by such mechanisms \mathcal{F} . As a shorthand, I will refer to a mechanism f or the set of

⁴This selection follows Holmstrom [1984]. Kamenica and Gentzkow [2011] use sender-preferred equilibrium as a related notion in information design.

mechanisms \mathcal{F} to refer to direct mechanisms that implement the respective choice rules.

Restricting attention to \mathcal{F} can not reduce the principal's utility. I can hence state her problem as follows:

$$\max_{f \in \mathcal{F}} E_{\mu} [f(\theta) \cdot v(\theta)]$$

2.4 Menus

A menu M is a convex, compact, non-empty subset of A . I will say that M has size n if $|Ext(M)| = n$. I will denote the set of all menus with \mathcal{M} . Consider the following indirect mechanism: The agent has a message space $S = M$, and the alternative corresponding to his message realizes.⁵ I will say that M is the menu of a direct mechanism $f \in \mathcal{F}$ if the allocation the agent receives under f is a best-response in this indirect mechanism.

Definition 2. *M is the menu of a mechanism $f \in \mathcal{F}$ if*

$$f(\theta) \in \arg \max_{a \in M} U(\theta, a).$$

In principle, a mechanism could have multiple menus, which is why at this point, to speak "the menu" is an abuse of language. Yet, I will demonstrate below that a unique menu is associated with every mechanism in our setting.

3 Extreme Points of the Mechanism Set

3.1 The Strong Equivalence between the Revelation and Taxation Principle

In this section, I characterize the extreme points of \mathcal{F} through the convex structure of \mathcal{M} . In particular, the set of menus has a convex structure

⁵This indirect approach to mechanism design is referred to as the taxation principle going back to Hammond [1979].

induced by Minkovski addition on sets. Hence the following is a strengthened version of equivalence between the revelation and taxation principles:

Theorem 1. *Define $T : \mathcal{F} \rightarrow \mathcal{M}$, s.t. $T(f) = \overline{\text{Conv}}(f(\Theta))$ for all $f \in \mathcal{F}$. Then T satisfies the following properties:*

- i T is a bijection that maps f to the menu of f .*
- ii $|f(\Theta)| = n < \infty$ if and only if $T(f)$ is a polytope with n vertices.*
- iii T preserves convex combinations, i.e., for all $f, f', f'' \in \mathcal{F}$*

$$f = \lambda f' + (1 - \lambda)f'' \iff T(f) = \lambda T(f') + (1 - \lambda)T(f'')$$

In particular, f is an extreme point of \mathcal{F} if and only if $T(f)$ is an extreme point of \mathcal{M} .

Proof. See Appendix A □

The first two points are the equivalence of revelation and taxation principle restated in the context of this model. In contrast, the third point shows that this equivalence preserves relevant convex structure. Therefore, it is worthwhile to view both convex combinations from the agent's perspective and view them as random mixing between different mechanisms. This interpretation on the direct side is straightforward. For the equivalence to hold, the agent must achieve the same overall selection if he selects from the Minkovski sum as if he were to select from both menus separately. However, by definition, the first choice is from all potential combinations, which is equivalent.

This result is the central connection between direct mechanisms and convex sets of probability distributions I employ in this paper.

3.2 Extreme Points of Menus

Extreme points of our menu set \mathcal{M} are such menus that are not the Minkovski sums of appropriately scaled feasible menus. Whether a given convex body

has a representation as the Minkovski sum of other convex bodies is a well-studied problem. Schneider [2014] is an excellent reference. The difference between both problems is the feasibility constraints for the sum and the summands present in the current setting.

To connect both problems and introduce the relevant notions, let me recall that two convex bodies $K, K' \in \mathbb{R}^d$ are homothetic if $K = \alpha K' + v$, for some $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^d$. Any convex body K can be written as the sum $K = (\lambda K - v) + ((1 - \lambda)K + v)$ for $\lambda \in (0, 1)$ and $v \in \mathbb{R}^d$. I will hence call such decompositions trivial.

Definition 3. *A convex body $K \subset \mathbb{R}^d$ is decomposable if it has a non-trivial decomposition, i.e., if there exist convex bodies $B, C \subset \mathbb{R}^d$, B and C not homothetic to K s.t. $K = B + C$. A convex body that is not decomposable is indecomposable.*

To compare the notion of $M \in \mathcal{M}$ being an extreme point of \mathcal{M} or being indecomposable, suppose there exist convex bodies B, C both different from M s.t. $M = B + C$. B and C are a counterexample to M being indecomposable if they are both not homothetic to M . In contrast B and C are a counterexample to M being an extreme point of \mathcal{M} if there exists a $\lambda \in (0, 1)$, s.t. $\frac{1}{\lambda}B$ and $\frac{1}{1-\lambda}C$ are both feasible menus, i.e. subsets of A , since then $M = \lambda \frac{1}{\lambda}B + (1 - \lambda) \frac{1}{1-\lambda}C$. In particular, no notion implies the other.

To connect the two concepts, I next discuss a notion of menus for which the constraint to be inside a given simplex is binding.

Definition 4. *A menu M is maximal if it consists of a single vertex of A or has a non-empty intersection with all facets of A .*

The next lemma establishes maximality as a necessary condition for a menu to be an extreme point.

Lemma 1. *A menu $M \in \mathcal{M}$ has a trivial decomposition into non-identical menus if and only if M is non-maximal.*

In particular, for such a menu there exists menus $M_h, M_d \in \mathcal{M}$ and $\lambda \in (0, 1)$, where $M_h \sim M$ and M and M_d consists of a single vertex of A s.t., $M = \lambda M_h + (1 - \lambda)M_d$.

Proof. See Appendix A. □

If $M \in \mathcal{M}$ is not maximal, it is fully enclosed in a smaller sub-simplex within A . It can be rewritten as a convex combination of a scaled-up original version and a vertex.

Note that the mechanism associated with a maximal menu either gives no choice to the agent and always implements the same outcome or grants a veto, i.e., the agent can ensure that a single outcome occurs with probability zero with a respective report. In particular, every mechanism with a range of size two maps to lotteries with disjoint supports.

There is a close connection between this result and the frequent observation that "there is no distortion at the top" in several models of mechanism design, such as the multi-object monopolistic seller problem. The usual argument for this observation is that the highest type does not grant information rents to any other type. However, a distortion at the top implies a trivial decomposition of the mechanism into a scaled-up version of the same mechanism and a mechanism that never allocates a given good, contradicting optimality.

$M \in \mathcal{M}$ being maximal is a necessary condition to be an extreme point. If, in addition, it is indecomposable, it has to be an extreme point since it neither has a trivial nor non-trivial decomposition into non-identical menus. In particular, it is not the sum of appropriately scaled feasible menus. It is not obvious that these conditions are also jointly necessary. For example, a menu might be decomposable, yet the scaled parts might not be feasible for any decomposition. This is the case when feasibility depends on being a subset of, e.g., a square. However, in this setting, being maximal and indecomposable characterizes extreme points.

Theorem 2. *A menu $M \in \mathcal{M}$ is an extreme point of \mathcal{M} if and only if M is maximal and indecomposable.*

Proof. See Appendix A. □

3.3 Extreme Points of the set of Direct Mechanisms

Theorem 1 and 2 jointly characterize the extreme points of \mathcal{F} . I summarize this below.

Theorem 3. *A non-constant mechanism $f \in \mathcal{F}$ is an extreme point of \mathcal{F} if it grants a veto and $T(f)$ is indecomposable.*

Proof. Immediate by Theorems 1 and 2. □

At this point, everything known about indecomposable convex bodies is a statement on extreme points in our model and vice versa. A full characterization of extreme points of direct mechanisms in terms other than these would substantially extend the existing geometry literature. However, no such attempt is made here.

Indecomposability is well understood in two dimensions and for polytopes in any dimension, which correspond to finite mechanisms. In particular, simple algebraic procedures exist to check whether a polytope is indecomposable by calculating the rank of an associated matrix. See Smilansky [1987] for details.

In the remainder of this section, I will focus on the consequences of two results on indecomposable sets.

As with the connection of granting a veto and no distortion at the top, both results have a substantially different yet closely related corresponding phenomenon in the multi-good monopolistic seller problem.

Delegation with one or two outcomes is each trivial. In the latter, the principal has to decide between taking the decision herself or full delegation. In contrast, the case with three alternatives allows intermediate extreme points, which still all have a simple structure.

Theorem 4. *Suppose $k \leq 3$. Then a non-constant mechanism f is an extreme point of \mathcal{F} if it grants a veto and $f(\Theta) \leq k$.*

Proof. See Appendix A □

Points, line segments, and triangles are the only indecomposable convex bodies in two dimensions.⁶

The case $k = 3$ corresponds to the one good case in the monopolistic seller problem. There are three possible "trades" of probabilities for two outcomes against a third. Any bundle of two trades is on net itself a single trade, even though potentially for a different "price" or rate of substitution.

The situation with four or more outcomes is in sharp contrast to this classification. It is similar to how bundling opportunities and possible discounts lead to a rich set of extreme points. In both cases, an additional lever to screen can be combined in a continuum of ways.

Theorem 5. *Suppose $k \geq 4$. Then the non-constant extreme points of \mathcal{F} with finite range are a dense subset of mechanisms granting a veto.*

Proof. See Appendix A □

There are two important consequences to this characterization. First, although extreme points can have menus of infinite sizes, any such extreme point is arbitrarily close to a mechanism with a finite menu size. Therefore it is approximately without loss to focus attention on this better-understood class. The second consequence, however, is that even this class is so rich that it is not an easily comprehensible and sufficient candidate class for optimization, as seen for $k = 3$.

4 Conclusion

I have characterized extreme points in a model of finite delegation, which subsumes several important models in mechanism design. The characterization builds the convex structure of the set of menus. My results are more general cases of previously observed phenomena in the multi-object monopolistic seller problem.

My characterization can be seen as a first step in analyzing multi-agent decentralization problems. For example, Börger and Postl [2009], and Kim

⁶The result was first mentioned in Gale [1954], yet no proof was published. It was later independently demonstrated by Meyer [1972] and Silverman [1973].

[2017] study a setting with two agents, three alternatives, and a benevolent principal maximizing total welfare. Similar problems have been analyzed via the reduced form approach. Yet this depends on a characterization of feasibility similar to the one achieved by Border [1991] for auctions. Gopalan et al. [2018] study such characterizations and find that a simple characterization for asymmetric agent models would contradict widely held beliefs in complexity theory. Nevertheless, due to fairness concerns in several applications, it seems reasonable that the symmetric case is of special relevance.

This paper may also contribute to understanding the relationship between delegation and Bayesian persuasion. Kolotilin and Zapechelnyuk [2019] and Kleiner et al. [2021] both find an equivalence between the two problems in one-dimensional cases. The present work might aid in understanding whether this equivalence breaks down in higher dimensions.

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A Appendix: Proofs

Proof of Theorem 1. i) Fix an arbitrary $\theta \in \Theta$, then

$$f(\theta) \in \arg \max_{a \in f(\Theta)} U(\theta, a) \subset \arg \max_{a \in \overline{Conv}(f(\Theta))} U(\theta, a)$$

Hence $T(f)$ is a menu of f .

Next, I show that an inverse function exists. For this, define f^M s.t.

$$f^M(\theta) \in \arg \max_{a \in M} U(\theta, a)$$

and f^M satisfies principle-preferred tie-breaking. Now $T^{-1} : \mathcal{M} \rightarrow \mathcal{F}$ with $T^{-1}(M) = f^M$ is the required inverse function by construction.

ii) If $|f(\Theta)|$ is finite, all extreme points of $T(f)$ are exposed. Hence $f(\Theta) = \text{Ext}(T(f))$.

iii) I will prove sufficiency first. For this I assume fix $f, f', f'' \in \mathcal{F}$ s.t. $f = \lambda f' + (1 - \lambda)f''$ for some $\lambda \in (0, 1)$. I can deduce the following:

$$\begin{aligned} T(f) &= T(\lambda f' + (1 - \lambda)f'') \\ &= \overline{Conv}(\lambda f'(\Theta) + (1 - \lambda)f''(\Theta)) \\ &= \overline{Conv}(\lambda f'(\Theta)) + \overline{Conv}((1 - \lambda)f''(\Theta)) \\ &= \lambda \overline{Conv}(f'(\Theta)) + (1 - \lambda) \overline{Conv}(f''(\Theta)) \\ &= \lambda T(f') + (1 - \lambda)T(f'') \end{aligned}$$

Now for the reverse direction fix $f, f', f'' \in \mathcal{F}$ s.t. $\overline{Conv}(f(\Theta)) = \lambda \overline{Conv}(f'(\Theta)) + (1 - \lambda) \overline{Conv}(f''(\Theta))$ for some $\lambda \in (0, 1)$. Then I can deduce:

$$\begin{aligned}
f &= T^{-1}(T(f)) \\
&= T^{-1}(\overline{Conv}(f(\Theta))) \\
&= T^{-1}(\lambda \overline{Conv}(f'(\Theta)) + (1 - \lambda) \overline{Conv}(f''(\Theta))) \\
&= \lambda T^{-1} \overline{Conv}(f'(\Theta)) + (1 - \lambda) T^{-1} \overline{Conv}(f''(\Theta)) \\
&= (\lambda T^{-1} T(f') + (1 - \lambda) T^{-1} T(f'')) \\
&= \lambda f' + (1 - \lambda) f''
\end{aligned}$$

□

Proof of Lemma 1. Suppose $M \in \mathcal{M}$ has an empty intersection with one facet. Without loss of generality, assume that for all $a = (a_1, \dots, a_k) \in M$, $a_1 \neq 0$. Since M is closed, there exists an $\varepsilon > 0$, s.t. $a_1 \geq \varepsilon$ for all $a \in M$. Define

$$M_\varepsilon = \{a_\varepsilon = (\frac{1}{1 - \varepsilon}a_1 - \varepsilon, \frac{1}{1 - \varepsilon}a_2, \dots, \frac{1}{1 - \varepsilon}a_k) | a \in M\}.$$

M_ε is a feasible menu since all probabilities in all alternatives are positive and add to 1. It is then easy to check that

$$M = \varepsilon(1, 0, \dots, 0) + (1 - \varepsilon)M_\varepsilon.$$

For the reverse, suppose $M \in \mathcal{M}$ is maximal and suppose

$$M = \lambda M' + (1 - \lambda)M''$$

for some $M', M'' \in \mathcal{M}$, s.t. $M \sim M' \sim M''$. If one of the parts does not intersect a facet, so does M , but since M is maximal, so are M', M'' . Yet this implies $M = M' = M''$.

□

Proof of Theorem 2. Suppose $M \in \mathcal{M}$ is indecomposable and maximal, and suppose there exists $M', M'' \in \mathcal{M}$, s.t.

$$M = \lambda M' + (1 - \lambda)M''$$

since M is indecomposable $M \sim M' \sim M''$, yet since M is maximal by Lemma 1: $M = M' = M''$.

Suppose M is an extreme point. Then M is maximal by Lemma 1. Suppose there exist convex bodies $K', K'' \in \mathbb{R}^d$, s.t. $M = K' + K''$.

There are unique maximal sets $M', M'' \in \mathcal{M}$, s.t. $M' \sim K'$ and $M'' \sim K''$. Therefore

$$M = \lambda M' + (1 - \lambda)M''$$

□

Proof of Theorem 4.

Theorem (Meyer [1972] and Silverman [1973]). *In \mathbb{R}^2 , an indecomposable compact convex set must be either a point, a line segment, or a triangle.*

□

Proof of Theorem 5.

Theorem (Shephard [1963]). *If all the 2-faces of a polytope P are triangles, then P is indecomposable.*

This set is dense in the set of convex bodies for the Hausdorff metric. See, e.g., Schneider [2014].

□