A Groundedness Predicate for Kripke's Theory of Truth

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Abstract

The object language of Kripke's 1975 semantic theory of truth, based on the Strong Kleene valuation scheme, cannot contain a predicate that expresses the notion "ungroundedness" that Kripke provides an analysis of. This is unfortunate; it means that, in the object language of Kripke's theory, there is no obvious way to express Kripke's diagnostic insight about what causes semantic pathology. This paper shows how to introduce a "groundedness" predicate, G, to a Kripkean theory of truth that can fill this expressive gap. In the fixed-point construction that gives an interpretation for G, G's anti-extension tracks networks of sentences that, due to predications of truth, result in non-terminating graphs of semantic dependence. In the fixed-point models that provide a class of intended interpretations for G: (i) every sentence with a classical semantic value is in the extension of G, (ii) every sentence in the anti-extension of G receives the value $\frac{1}{2}$, and (iii) the anti-extension of G includes *all* the sentences that receive $\frac{1}{2}$ in the corresponding model of Kripke's original theory. A language augmented with predicates for truth and groundedness possesses sufficient expressive resources to articulate Kripke's diagnostic insight as it applies to itself.

It is well-known that the object language described by Kripke's 1975 semantic theory of truth, based on the Strong Kleene valuation scheme, cannot contain a predicate/operator that expresses "ungroundedness." This would be an object language predicate that, relative to a given fixed-point model, is true of all sentences that do not receive a classical truth-value in that model. The introduction of such a predicate would undermine the monotonicity of the jump operation used to construct Kripke's fixed-point models. The simplest way to see this is to note that the resulting language would contain Strengthened Liar sentences like (1):

(1) (1) is not true or (1) is ungrounded.

There is no fixed-point model based on the Strong Kleene scheme in which (1) is in the extension of "ungrounded" if and only if it does not receive a classical value. Given the intended interpretation of "ungrounded," it is inconsistent to suppose that (1) doesn't have a classical value, and any model in which (1) does have a classical value cannot be a fixed-point.

This aspect of Kripke's theory is unfortunate. As Hartry Field puts it, we would like to be able to express in the object language, in an assertive way, that pathological sentences like the Liar (2008 pg. 94-96) are to be *rejected.* If there is no predicate in the object language that expresses ungroundedness, it is not clear how we can do this. Moreover, Kripke's semantic theory has struck many as offering an insightful *diagnosis* of Liar-like pathology, and it would be nice to be able to express this in the object language as well. That diagnosis, roughly, is that sentences that predicate truth semantically *depend* on the sentences they predicate truth of, and Liar-like pathology is caused by being part of a non-well-founded network of semantic dependence. Since the object language of Kripke's theory cannot contain a predicate that holds just of the sentences that are semantically ungrounded in this way, it would appear that there are no resources in the object language for expressing Kripke's diagnostic insight. Part of the attraction of Kripke's theory rests on the fact that it makes intelligible how we can talk, in the object language, about the truth of object language sentences, so it is, *prima facie*, a defect if so much of what is philosophically interesting regarding Kripke's response to the semantic *paradoxes* can only be expressed once we make the reflective ascent to a metalanguage.

I'll suggest in this paper that we can develop a Kripkean theory of truth that goes a long way to satisfying these desiderata if we relax our standard for what counts as an ungroundedness predicate. Arguably, the intended interpretation of the "ungroundedness" predicate described above—a predicate that is true of all sentences that do not receive a classical value—is stronger than it is reasonable to demand. It would not be so surprising if a groundedness/ungroundedness predicate is susceptible to semantic failure for similar reasons that a truth/falsity predicate is. And if so, we might expect that, for some objects that are ungrounded, applying the groundedness/ungroundedness predicate to those objects just results in semantic failure rather than a claim with a truth-value.

Here I develop a fixed-point semantics for a language $\mathcal{L}$[T,G] that extends the usual language for a Kripkean theory of truth, $\mathcal{L}$[T], by adding a *Groundedness* predicate G. In the fixed-point models that will be my focus, G has the following properties:

- Every sentence with a classical truth-value in that model is in the extension of G.
- Every sentence in the anti-extension of G receives the value $\frac{1}{2}$ in that model.
- The anti-extension of G in that model includes *all* the sentences that receive $\frac{1}{2}$ in

the corresponding $\mathcal{L}$[T] model (of Kripke's original theory).

I begin, in Section 1, by introducing the language $\mathcal{L}$[T,G] and defining the notions relevant to the jump operation by which I construct fixed-point models. In Section 2, following Kripke, I define the sequences that lead to the construction of such models. In Section 3, I compare my $\mathcal{L}$[T,G] models to the corresponding models of Kripke's original account. The main result of this section is Theorem 16: for all sentences p of $\mathcal{L}$[T], p receives the value $\frac{1}{2}$ in an $\mathcal{L}$[T] fixed-point if and only if p is in the anti-extension of "G" in the corresponding $\mathcal{L}$[T,G] fixed-point. I close, in Section 4, by elaborating on my suggestion that, by introducing the G predicate, we end up with a language in which we can affirmatively express rejection of semantically pathological sentences, and can express the diagnostic insight of Kripke's theory of truth. I also compare my G predicate with other proposed predicates and operators that have been introduced to address similar expressive limitations in Kripke's theory of truth.

1 Definitions

Let $\mathcal{L}$ be a first-order language containing only $\neg, \vee$, and the existential quantifier as logical operators. Let $\mathcal{L}$[T] be $\mathcal{L}$ with an additional unary predicate, T, the truth predicate, as well as a set of "distinguished constants" $A = \{a_0, a_1, a_2, ...\}$ with the same cardinality as the natural numbers. These constants will function as names for the sentences of $\mathcal{L}$[T]. They are distinguished in the sense that their denotation will be held fixed in all of the models under consideration; their denotation is given by a function f_1: $A \to \{\phi \mid \phi \text{ is a sentence of } \mathcal{L}[\mathrm{T}]\}$. This provides us with a mechanism for referring to $\mathcal{L}$[T] sentences within $\mathcal{L}$[T].

Let $\mathcal{L}$[T,G] be $\mathcal{L}$[T] with an additional unary predicate G, the groundedness predicate, and a further set of distinguished constants $B = \{b_0, b_1, b_2, ...\}$ with the same cardinality as the natural numbers. The denotation of the members of B will be held fixed across models and it is given by a function f_2 such that f_2: $A \cup B \to \{\phi \mid \phi \text{ is a sentence of } \mathcal{L}[\mathrm{T,G}]\}$, where for every i, $f_1(a_i) = f_2(a_i)$. Again, f_2 provides a means for referring to $\mathcal{L}$[T,G] sentences in $\mathcal{L}$[T,G]. Where ϕ is a sentence of $\mathcal{L}$[T] or $\mathcal{L}$[T,G], I will use the notation $< \phi >$ for the distinguished name $c \in A \cup B$ such that $f_2(c) = \phi$.[1] I will call any sentence of $\mathcal{L}$[T,G] that contains an explicit occurrence of T an **alethic** sentence.

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[1] This is the approach to self-reference adopted in Barwise and Etchemendy 1987; Ripley 2012; Tourville and Cook 2016, 2020; Szmuc and Rosenblatt 2014; Gallovich and Rosenblatt 2024. A consequence of this method is that self-reference in $\mathcal{L}$[T] and $\mathcal{L}$[T,G] is not a purely syntactic matter. This contrasts with the more typical approach in formal theories of truth where one works with an arithmetic language and emulates self-reference via Gödel-coding. It's possible to carry out the fixed-point constructions I describe in a language that extends arithmetic, but, for reasons I'll point out shortly, doing so will introduce some technical complexities to the comparison of $\mathcal{L}$[T] and $\mathcal{L}$[T,G].

An $\mathcal{L}[\mathrm{T}]$ model is an ordered pair $\mathcal{M} = \langle I, \langle \mathrm{T}_+, \mathrm{T}_- \rangle\rangle$, where I is a standard classical interpretation of $\mathcal{L}$, T_+ is the extension of T, T_- is the anti-extension of T. These are versions of the K3 models familiar from Kripke 1975.

I will assume that three restrictions hold for the domain of every $\mathcal{L}[\mathrm{T}]$ model: (i) every element in the domain is the denotation of some term in $\mathcal{L}[\mathrm{T}]$, (ii) the domain includes every sentence of $\mathcal{L}[\mathrm{T}]$, and (iii) the domain does not include any sentence of $\mathcal{L}[\mathrm{T,G}]$ that is not a sentence of $\mathcal{L}[\mathrm{T}]$. The first restriction is dispensable—it allows for a simplified presentation of the semantics of quantified sentences.[2] The second restriction is required to ensure that the interpretation of the distinguished constants in A can be held fixed for all $\mathcal{L}[\mathrm{T}]$ models. The third restriction has the following rationale. Any semantic theory of truth for a given language needs to make choices regarding the treatment of predications of truth to objects that are not sentences of that language. For instance, Tarski 1956 and Kripke 1975 both treat all such predications as false. In the present context, where we are comparing the two languages $\mathcal{L}[\mathrm{T}]$ and $\mathcal{L}[\mathrm{T,G}]$, this treatment will produce unwanted incongruities. For instance, if T(c) is an $\mathcal{L}[\mathrm{T}]$ sentence where c denotes an $\mathcal{L}[\mathrm{T,G}]$ sentence that is not in $\mathcal{L}[\mathrm{T}]$, we do not want the truth-value of this sentence to potentially vary between $\mathcal{L}[\mathrm{T}]$ models and $\mathcal{L}[\mathrm{T,G}]$ models simply because c denotes a sentence that is not in $\mathcal{L}[\mathrm{T}]$. To avoid this problem, I am adopting the brute-force approach of stipulating that no term in $\mathcal{L}[\mathrm{T}]$ denotes any sentence of $\mathcal{L}[\mathrm{T,G}]$. For the purposes of this paper, $\mathcal{L}[\mathrm{T}]$ is mainly used as a foil for assessing whether $\mathcal{L}[\mathrm{T,G}]$ expresses Kripke's diagnostic insight, so nothing substantive depends on the restriction.[3] What I am calling Kripke's diagnostic insight is orthogonal to the question of how to handle predications of truth, in $\mathcal{L}[\mathrm{T}]$, to non-$\mathcal{L}[\mathrm{T}]$ sentences.

I recursively define valuation functions on $\mathcal{L}[\mathrm{T}]$ models by first characterizing the denotation of $\mathcal{L}[\mathrm{T}]$ terms relative to those models.

Definition 1 *For any $\mathcal{L}[\mathrm{T}]$ model $\mathcal{M}[\mathrm{T}]$, the denotation function of that model $\delta(\mathcal{M}[\mathrm{T}])$ is a function whose domain is the set of terms of $\mathcal{L}[\mathrm{T}]$ such that, for any term t in $\mathcal{L}[\mathrm{T}]$:*

(i) if $t \notin A$, $\delta(\mathcal{M}[\mathrm{T}])(t) = I(t)$

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[2] Dropping this assumption and introducing variable assignment functions (or some equivalent device) would require corresponding adjustments to the definitions of *direct call* and *settled-as-true* (Definitions 5 and 8)—i.e. direct calling would have to be defined to hold between pairs of sentences and assignments. These additional complications would not, I think, affect the substance of my account.

[3] There are less brutish ways of solving this problem, but they involve other trade-offs in complexity. For instance, we could follow Cook 2007 in using Gödel-coding as our device of self-reference, stipulate that the coding scheme is held fixed for $\mathcal{L}[\mathrm{T}]$ and $\mathcal{L}[\mathrm{T,G}]$, and introduce a placeholder value **n** to $\mathcal{L}[\mathrm{T}]$ models such that any sentence of the form $\mathrm{T}(m)$, where m is a numeral that codes a sentence that is not in $\mathcal{L}[\mathrm{T}]$, receives **n**. This would ensure that the problem I allude to above does not arise. It would, however, mean that all results from Lemma 8 to Corollary 12.1 would need an extra proviso restricting them to sentences of $\mathcal{L}[\mathrm{T}]$ *that do not receive* **n**, and a number of the proofs below would need to be extended to address such sentences. And it is, itself, a somewhat unfamiliar departure from Kripke's original approach.

(ii) if $t \in A$, $\delta(\mathcal{M}[\mathrm{T}])(t) = f_1(t)$

Now we can define the valuation function of an $\mathcal{L}[\mathrm{T}]$ model $\mathcal{M}[\mathrm{T}]$.

Definition 2 *If* $\mathcal{M}[\mathrm{T}]$ *is an* $\mathcal{L}[\mathrm{T}]$ *model, then the valuation of* $\mathcal{M}[\mathrm{T}]$, $v(\mathcal{M}[\mathrm{T}])$, *is a function from* $\mathcal{L}[\mathrm{T}]$ *sentences to* $\{0, \frac{1}{2}, 1\}$ *such that, for any sentence* ϕ *of* $\mathcal{L}[\mathrm{T}]$*:*

(i) if ϕ *is a sentence of* $\mathcal{L}$*, then* $v(\mathcal{M}[\mathrm{T}])(\phi) = I(\phi)$
(ii) if ϕ *is T(c), for some term c, then:*

- *if the denotation of* $c \in T_+$, $v(\mathcal{M}[\mathrm{T}])(\phi) = 1$
- *if the denotation of* $c \in T_-$, $v(\mathcal{M}[\mathrm{T}])(\phi) = 0$
- *otherwise,* $v(\mathcal{M}[\mathrm{T}])(\phi) = \frac{1}{2}$

(iii) $v(\mathcal{M}[\mathrm{T}])(\neg\phi) = 1 - v(\mathcal{M}[\mathrm{T}])(\phi)$
(iv) $v(\mathcal{M}[\mathrm{T}])(\phi \vee \psi) = \max\{v(\mathcal{M}[\mathrm{T}])(\phi), v(\mathcal{M}[\mathrm{T}])(\psi)\}$
(v) $v(\mathcal{M}[\mathrm{T}])(\exists v \phi(v)) = \max\{v(\mathcal{M}[\mathrm{T}])(\phi(c/v)) \mid c \text{ is a term of } \mathcal{L}[\mathrm{T}]\}$

In the last case, $\phi(c/v)$ indicates the result of uniformly substituting c for v in ϕ. This amounts to treating existential generalization as a form of infinite disjunction.

Now I give the parallel definitions for $\mathcal{L}$[T,G] models. An $\mathcal{L}$[T,G] model is a triple $\mathcal{M} = \langle I, \langle \mathrm{T}_+, \mathrm{T}_- \rangle, \langle \mathrm{G}_+, \mathrm{G}_- \rangle \rangle$, where I is a standard classical interpretation of $\mathcal{L}$, T_+ is the extension of T, T_- is the anti-extension of T, G_+ is the extension of G, and G_- is the anti-extension of G. I assume that (i) the domain of every $\mathcal{L}$[T,G] model includes every sentence of $\mathcal{L}$[T,G] and (ii) every element in the domain is the denotation of some term in $\mathcal{L}[T, G]$.

Definition 3 *For any* $\mathcal{L}$[T,G] *model* $\mathcal{M}$[T,G]*, the denotation function of that model* $\delta(\mathcal{M}[\mathrm{T,G}])$ *is a function whose domain is the set terms of* $\mathcal{L}$[T,G] *such that, for any term t in* $\mathcal{L}$[T,G]*:*

(i) if $t \notin A \cup B$, $\delta(\mathcal{M}[\mathrm{T,G}])(t) = I(t)$
(ii) if $t \in A \cup B$, $\delta(\mathcal{M}[\mathrm{T,G}])(t) = f_2(t)$

Definition 4 *If* $\mathcal{M}$[T,G] *is an* $\mathcal{L}$[T,G] *model, then the valuation of* $\mathcal{M}$[T,G], $v(\mathcal{M}[\mathrm{T,G}])$, *is a function from* $\mathcal{L}$[T,G] *sentences to* $\{0, \frac{1}{2}, 1\}$ *such that, for any sentence* ϕ *of* $\mathcal{L}$[T,G]*:*

(i) if ϕ *is a sentence of* $\mathcal{L}$*, then* $v(\mathcal{M}[\mathrm{T,G}])(\phi) = I(\phi)$
(ii) if ϕ *is T(c), for some term c, then:*

- *if the denotation of* $c \in T_+$, $v(\mathcal{M}[\mathrm{T,G}])(\phi) = 1$
- *if the denotation of* $c \in T_-$, $v(\mathcal{M}[\mathrm{T,G}])(\phi) = 0$
- *otherwise,* $v(\mathcal{M}[\mathrm{T,G}])(\phi) = \frac{1}{2}$

(iii) if ϕ is G(c), for some term c, then:

- *if the denotation of* $c \in G_+$, $v(\mathcal{M}[\mathrm{T,G}])(\phi) = 1$
- *if the denotation of* $c \in G_-$, $v(\mathcal{M}[\mathrm{T,G}])(\phi) = 0$
- *otherwise,* $v(\mathcal{M}[\mathrm{T,G}])(\phi) = \frac{1}{2}$

(iv) $v(\mathcal{M}[\mathrm{T,G}])(\neg\phi) = 1 - v(\mathcal{M}[\mathrm{T,G}])(\phi)$
(v) $v(\mathcal{M}[\mathrm{T,G}])(\phi \vee \psi) = \max\{v(\mathcal{M}[\mathrm{T,G}])(\phi), v(\mathcal{M}[\mathrm{T,G}])(\psi)\}$
(vi) $v(\mathcal{M}[\mathrm{T,G}])(\exists v\phi(v)) = \max\{v(\mathcal{M}[\mathrm{T,G}])(\phi(c/v)) \mid c \text{ is a term of } \mathcal{L}[\mathrm{T,G}]\}$

Before proceeding, I'll introduce a couple of notational abbreviations for talking about models. A simplified representation of a model $\mathcal{M}$ for $\mathcal{L}[\mathrm{T,G}]$ is $\mathcal{M}[\mathrm{T,G}]$, where T abbreviates $\langle \mathrm{T}_+, \mathrm{T}_- \rangle$, the "truth concept" of $\mathcal{M}$, and G abbreviates $\langle \mathrm{G}_+, \mathrm{G}_- \rangle$, the "groundedness concept" of $\mathcal{M}$. A truth concept T$'$ **extends** a truth concept T iff $\mathrm{T}_+ \subseteq \mathrm{T}'_+$ and $\mathrm{T}_- \subseteq \mathrm{T}'_-$. Likewise, a groundedness concept G$'$ extends a groundedness concept G iff $\mathrm{G}_+ \subseteq \mathrm{G}'_+$ and $\mathrm{G}_- \subseteq \mathrm{G}'_-$. A model $\mathcal{M}[\mathrm{T}', \mathrm{G}']$ extends a model $\mathcal{M}[\mathrm{T,G}]$ iff (a) T$'$ extends T and G$'$ extends G and (b) the background interpretation I of the two models is the same.

Our ultimate goal is to describe a G predicate whose anti-extension tracks sentences that can be shown to be members of an illegitimate semantic dependence structure. This sort of illegitimate structure is given an analysis in the definition of an **ungrounded set**, Definition 9. I'll start by explaining some notions that are presupposed by that definition: **direct calling**, **dependence graphs**, and a model's **settling the truth-value** of a sentence.

Direct calling is the relation of semantic dependence that is at the heart of Kripke's analysis of ungroundedness. I'll use the notation R for this relation.[4]

Definition 5 *R is a binary relation on* $\mathcal{L}[\mathrm{T,G}]$ *such that, for any sentences x and y, Rxy iff one of the following holds:*

(i) x is $T(<\phi>)$ *and y is* ϕ
(ii) x is a sentence $\neg\phi$ *and y is* ϕ
(iii) x is $\phi \vee \psi$ *and y is* ϕ *or* ψ
(iv) x is $\exists v\phi$ *and y is* $\phi[v/c]$ *for some constant c, where* $\phi[v/c]$ *is* ϕ *with every free occurrence of v replaced with c*

Intuitively, direct calling is a form of semantic dependence. If x directly calls x, then the determination of a truth-value for x depends on a prior determination of a truth-value for y. Complex sentences directly call their subsentences, quantified sentences directly

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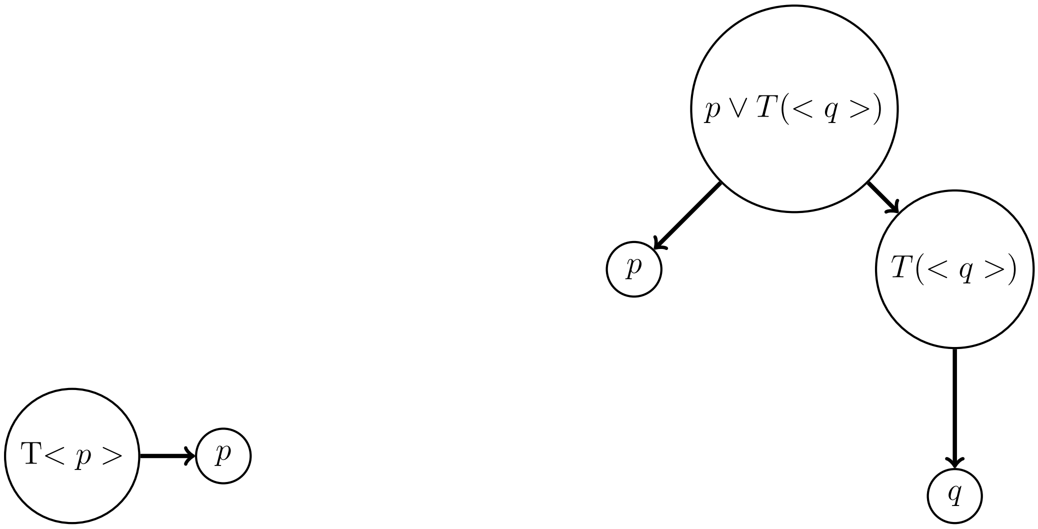
[4] I borrow this terminology from Gaifman 1988, 1992, 2000. For similar graph-theoretic treatments of semantic dependence, see Yablo 1982 and Cook 2004.



(a) Two-membered dependence graph (b) Four-membered dependence graph

Figure 1: Dependence graphs

call their instances, and any sentence of the form T($< \phi >$) directly calls ϕ. A sequence of $\mathcal{L}$[T,G] sentences $p_0, \ldots, p_n$ is a **calling path** iff, for every consecutive pair in the sequence $< x, y >$, Rxy or Ryx. Note: predications of G do not initiate direct call relations. So a sentence of the form G($< \phi >$) does not call ϕ.

Definition 6 *If W is a set of $\mathcal{L}$[T,G] sentences, then a directed graph $< R, W >$ is a **dependence graph** iff, for all x and y in W, there is a calling path $p_0, \ldots, p_n$, where $x = p_0$ and $y = p_n$.*

A dependence graph is a collection of sentences that are all related to each other via calling paths. So for instance, supposing that p and q are $\mathcal{L}$[T,G] sentences, the diagrams in Figure 1 represent dependence graphs. Each arrow in the diagram is an instance of the R relation and each circle is an $\mathcal{L}$[T,G] sentence. These diagrams satisfy the definition of a dependence graph, because, in each diagram, there is a calling path from each circle to every other circle. A **terminal node** in a dependence graph $< R, W >$ is a sentence $x \in W$ such that $\neg\exists y(y \in W \wedge Rxy)$, i.e. a sentence that doesn't directly call any member of W. So, for instance, in 1(a) p is the only terminal node, and in 1(b) p and q are the only terminal nodes.

The anti-extension of the G predicate will target sets of sentences that constitute malformed dependence graphs. To explain the relevant sense of "malformed," we first need to explain what it is for an $\mathcal{L}$[T,G] model to settle the truth-value of a sentence:

Definition 7 *We inductively define **Settled-as-true** relative to a model $\mathcal{M}$ and **Settled-as-false** relative to a model $\mathcal{M}$ for sentences of $\mathcal{L}$[T,G]:*

If p is an atomic sentence, then p is settled-as-true relative to a model $\mathcal{M}$ iff:
(i) $v(\mathcal{M})(p) = 1$, or
(ii) p is $T(<\phi>)$ and $v(\mathcal{M})(\phi) = 1$.

If p is an atomic sentence, then p is settled-as-false relative to a model $\mathcal{M}$ iff:
(i) $v(\mathcal{M})(p) = 0$, or
(ii) p is $T(<\phi>)$ and $v(\mathcal{M})(\phi) = 0$

If p is a complex sentence, then p is settled-as-true relative to a model $\mathcal{M}$ iff:
(i) p is $\neg\phi$ and ϕ is settled-as-false in $\mathcal{M}$.
(ii) p is $\phi \vee \psi$ and either ϕ or ψ is settled-as-true in $\mathcal{M}$.
(iii) p is $\exists v \phi$ and some sentence $\phi[v/c]$ is settled-as-true in $\mathcal{M}$.

If p is a complex sentence, then p is settled-as-false relative to a model $\mathcal{M}$ iff:
(i) p is $\neg\phi$ and ϕ is settled-as-true in $\mathcal{M}$.
(ii) p is $\phi \vee \psi$ and both ϕ and ψ are settled-as-false on $\mathcal{M}$.
(iii) p is $\exists v \phi$ and all sentences $\phi[v/c]$ are settled-as-false in $\mathcal{M}$.

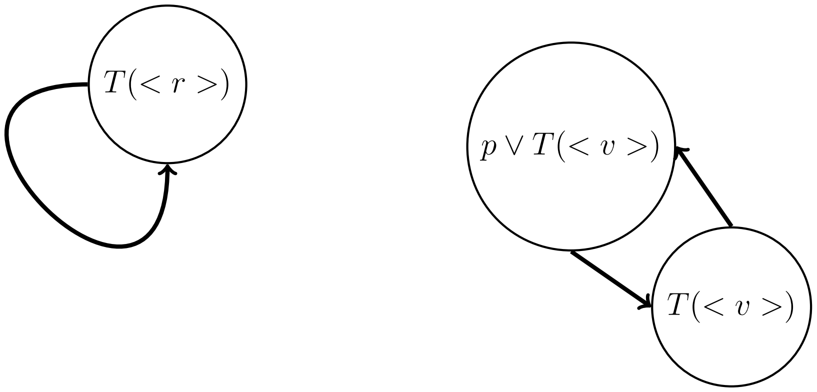
Definition 8 *For any sentence x of $\mathcal{L}[T, G]$, x is* ***settled*** *by a model $\mathcal{M}$ iff x is settled-as-true in $\mathcal{M}$ or settled-as-false in $\mathcal{M}$.*

Now we can give an analysis of the sort of semantic pathology that the anti-extension of the G predicate is intended to track:

Definition 9 *A set of $\mathcal{L}$[T,G] sentences B is an* ***ungrounded set*** *relative to a model $\mathcal{M}$ iff all of the following hold:*
(i) No member of B is settled by $\mathcal{M}$
(ii) for all x, if $v(\mathcal{M})(x) = \frac{1}{2}$ and some member of B directly calls x, then $x \in B$ (i.e. B is closed under direct call among the sentences that receive $\frac{1}{2}$ in $\mathcal{M}$)
(iii) the dependence graph $< R, B >$ has no terminal nodes

I'll illustrate this definition with two examples in Figure 2. In Figure 2(a), suppose that r is the sentence $T(< r >)$, and in Figure 2(b), suppose that v is the sentence $p \vee T(< v >)$.

Suppose that $\mathcal{M}$ is an $\mathcal{L}$[T,G] model such that $v(\mathcal{M})(r) = \frac{1}{2}$. Then $\{r\}$ will be an ungrounded set relative to $\mathcal{M}$. Since $v(\mathcal{M})(r) = \frac{1}{2}$ and r is T($< r >$), it is not settled as true or as false by $\mathcal{M}$. Further, the only sentence r directly calls is itself. $\{r\}$ is therefore closed under direct call and, *a fortiori*, closed under direct call among sentences that receive $\frac{1}{2}$ in $\mathcal{M}$. Finally, the dependence graph depicted in Figure 2(a), $< R, \{r\} >$, has no terminal nodes—the calling path emerging from r simply goes in a loop.



(a) Truth-teller

(b) Disjunctive Truth-Teller

Figure 2: Possible Ungrounded Sets

r is a truth-teller sentence, and the last paragraph gives a diagnosis of what goes wrong with such sentences. r is such that there cannot be any explanation of its truth or falsity. Since it only semantically depends on itself, the only thing that could explain its truth-value is its own truth-value.[5] Intuitively, this sort of explanatory structure is illegitimate—explanations cannot go in circles or continue without end. If x is a member of an ungrounded set relative to a model $\mathcal{M}$, then all of the terminating paths of x's semantic dependence graph have been evaluated, and there is still no basis for determining x's truth-value. According to what I'm calling Kripke's diagnostic insight, being a member of such an ungrounded set is the source of semantic pathology.

The set $Z = \{p \vee \mathrm{T}(< v >), \mathrm{T}(< v >)\}$ is a more interesting case, because whether or not it constitutes an ungrounded set relative to a model $\mathcal{M}$ depends not just on the values $\mathcal{M}$ gives to the members of Z but also on the value it gives p. In the case where $v(\mathcal{M})(p) = 0$ and each of the members of Z receive $\frac{1}{2}$, Z is an ungrounded set on $\mathcal{M}$. In that case, the members of Z are a sort of truth-telling loop. But if $v(\mathcal{M})(p) = 1$, then Z is not an ungrounded set on $\mathcal{M}$. For, in that case, Definitions 4 and 7 imply that both members of Z are settled-as-true in $\mathcal{M}$. And in the case where p and each member of Z all receive $\frac{1}{2}$ in $\mathcal{M}$, Z is not an ungrounded set either. In that case, Z is not closed under direct call among sentences that receive $\frac{1}{2}$ in $\mathcal{M}$. Z may be a subset of a larger ungrounded set in $\mathcal{M}$, but that will depend on further facts about p and what direct calls it initiates.

I'll note two important facts about ungrounded sets. First, if H is an ungrounded set relative to $\mathcal{M}$, then every sentence $p \in H$ is such that $v(\mathcal{M})(p) = \frac{1}{2}$. This holds because, by Definitions 8 and 9, every sentence which is not settled by a model $\mathcal{M}$ receives $\frac{1}{2}$ in $\mathcal{M}$. The second fact I will introduce as Lemma 1.

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[5] Of course, in a model where r receives a classical value, {r} will not be an ungrounded set. And there are fixed-point models for $\mathcal{L}$[T,G] where r does receive a classical value. These are models in which r has a truth-value though there is no explanation why.

Lemma 1 *For any* $\mathcal{L}$[T,G] *sentence* p *and any* $\mathcal{L}$[T,G] *model* $\mathcal{M}$*, if* p *is a member of an ungrounded set in* $\mathcal{M}$*, then* p *is an alethic sentence, i.e.* p *contains an occurrence of* T*.*

The proof of this lemma is an induction on formula complexity that I leave to the reader. The only way for a dependence graph $< R, X >$ to be such that none of its nodes are terminal is for every member of X to contain an occurrence of the truth-predicate. (This is borne out by the examples in Figure 2.)

Now I'll define the operation by which sentences get "added" to the extension and anti-extension of G. This is a function $J^\dagger$ from $\mathcal{L}$[T,G] models to $\mathcal{L}$[T,G] models, based on Kripke's jump operator. This operation leaves the background interpretation I unchanged. For the following three definitions, let Q be the set of all objects in the domain of the relevant models that are not sentences of $\mathcal{L}$[T,G].

Definition 10 $J^\dagger$ *is a function from* $\mathcal{L}$[T,G] *models to* $\mathcal{L}$[T,G] *models.* $J^\dagger(\mathcal{M}[\mathrm{T,G}]) = \mathcal{M}[\mathrm{T}^{'}, \mathrm{G}^{'}]$*, where:*

$$\mathrm{T}^{'}_{+} = \{x \mid v(\mathcal{M}[\mathrm{T,G}])(x) = 1\}$$
$$\mathrm{T}^{'}_{-} = \{x \mid v(\mathcal{M}[\mathrm{T,G}])(x) = 0\} \cup Q$$
$$\mathrm{G}^{'}_{+} = \{x \mid v(\mathcal{M}[\mathrm{T,G}])(x) = 1 \text{ or } v(\mathcal{M}[\mathrm{T,G}])(x) = 0\}$$
$$\mathrm{G}^{'}_{-} = \mathrm{G}_{-} \cup \{x \mid x \text{ is a member of an ungrounded set in } \mathcal{M}[\mathrm{T,G}]\}$$

The extension of T in $J^\dagger(\mathcal{M}[\mathrm{T,G}])$ is the set of all sentences that receive 1 in $\mathcal{M}$[T,G]; the anti-extension of T is the set of all non-sentences together with all the sentences that receive 0 in $\mathcal{M}$[T,G];[6] the extension of G is the set of all sentences that receive a classical value in $\mathcal{M}$[T,G]; and the new anti-extension of G collects all the members of the old anti-extension together with all the members of ungrounded sets in $\mathcal{M}$[T,G].

Since I ultimately want to compare my construction based on $\mathcal{L}$[T,G] models to Kripke's original construction, I'll also, in parallel, provide definitions for the key notions involved in the latter. I call Kripke's jump operator J^*.

Definition 11 J^* *is a function from* $\mathcal{L}$[T] *models to* $\mathcal{L}$[T] *models.* $J^*(\mathcal{M}[\mathrm{T}])) = \mathcal{M}[\mathrm{T}^{'}]$*, where:*

$$\mathrm{T}^{'}_{+} = \{x \mid v(\mathcal{M}[\mathrm{T}])(x) = 1\}$$
$$\mathrm{T}^{'}_{-} = \{x \mid v(\mathcal{M}[\mathrm{T}])(x) = 0\} \cup Q$$

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[6] Here I'm following Kripke 1975 in taking all non-sentences to be untrue. This is, I think, philosophically reasonable and allows for a familiar and simple presentation of the fixed-point construction. Though, as Kremer 1988 points out, it is a choice-point. Whether one assumes that non-sentences are true, false or neither, one can carry out Kripke's construction substantially unchanged. My fixed-point construction and the resulting correspondence between $\mathcal{L}$[T,G] and $\mathcal{L}$[T] fixed-points can likewise be developed in a setting where predications of truth to non-sentences are evaluated as $\frac{1}{2}$ or a setting where no assumptions about their valuation are made at all. Doing so, however, requires some technical adjustments to Definitions 10-12 and the Monotonicity theorems of Section 3.

Again following Kripke, I will be considering fixed-point models that are generated by a sequence starting with a certain base model. The construction can only be carried out if we make certain assumptions about that base model—namely that it is in **good standing**, in the sense defined below.

Definition 12 *An* $\mathcal{L}[\mathrm{T}]$ *model* $\mathcal{M}$ *is in* ***good standing*** *iff for every sentence* ϕ *of* $\mathcal{L}[\mathrm{T}]$*:*

(i) if $v(\mathcal{M})(T(<\phi>)) \in \{0,1\}$, *then* $v(\mathcal{M})(\phi) = v(\mathcal{M})(T(<\phi>))$, *and*

(ii) $Q \subseteq \mathrm{T}_-$

An $\mathcal{L}[\mathrm{T,G}]$ *model* $\mathcal{M}$ *is in* ***good standing*** *iff: for every sentence* ϕ *of* $\mathcal{L}[\mathrm{T,G}]$

(i) if $v(\mathcal{M})(T(<\phi>)) \in \{0,1\}$, *then,* $v(\mathcal{M})(\phi) = v(\mathcal{M})(T(<\phi>))$ *and* $v(\mathcal{M})(G(<\phi>)) = 1$, *and*

(ii) $Q \subseteq \mathrm{T}_-$, *and*

(iii) $\mathrm{G}_- = Q$

Finally, in order to compare the two constructions it will be necessary to define the sense in which an $\mathcal{L}[\mathrm{T}]$ model and an $\mathcal{L}[\mathrm{T,G}]$ model can be said to correspond to each other.

Definition 13 *An* $\mathcal{L}[\mathrm{T}]$ *model* $\mathcal{M} =< I, \mathrm{T} >$ *and an* $\mathcal{L}[\mathrm{T,G}]$ *model* $\mathcal{M}' =, < I', \mathrm{T}', \mathrm{G} >$ ***correspond to each other*** *iff:*

(i) Removing $\{x \,|\, x \text{ is a sentence of } \mathcal{L}[\mathrm{T,G}] \text{ that is not in } \mathcal{L}[\mathrm{T}]\}$ *from the domain of* I' *yields* I

(ii) $\mathrm{T} = \mathrm{T}'$

If an $\mathcal{L}[\mathrm{T}]$ model and $\mathcal{L}[\mathrm{T,G}]$ model correspond to each other, then their truth-concepts are the same and their background interpretation is the same, except that the domain of the latter contains $\mathcal{L}[\mathrm{T,G}]$ sentences that are not $\mathcal{L}[\mathrm{T}]$ sentences.

2 The † construction

The next definition describes the construction that yields fixed-point models for $\mathcal{L}[\mathrm{T,G}]$.[7]

Definition 14 *Suppose that* $\mathcal{M}[\mathrm{T,G}]$ *is an* $\mathcal{L}[\mathrm{T,G}]$ *model that is in good standing. We define a sequence of* $\mathcal{L}[\mathrm{T,G}]$ *models as follows:*

(i) $\mathcal{M}[\mathrm{T,G}]_0 = \mathcal{M}[\mathrm{T,G}]$

(ii) $\mathcal{M}[\mathrm{T,G}]_{\alpha+1} = J^{\dagger}(\mathcal{M}[\mathrm{T,G}]_{\alpha})$

(iii) Where α *is a limit ordinal,* $\mathcal{M}[\mathrm{T,G}]_{\alpha} = \mathcal{M}[\langle \bigcup_{\beta<\alpha} \mathrm{T}_+^{\beta}, \bigcup_{\beta<\alpha} \mathrm{T}_-^{\beta} \rangle, \langle \bigcup_{\beta<\alpha} \mathrm{G}_+^{\beta}, \bigcup_{\beta<\alpha} \mathrm{G}_-^{\beta} \rangle]$

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[7] The organization of the proofs in this section draws on unpublished notes by Sean Walsh and Salvatore Florio.

(In the last clause T_+^β refers to the T_+ of $\mathcal{M}[\mathrm{T,G}]_\beta$, T_-^β refers to the T_- of $\mathcal{M}[\mathrm{T,G}]_\beta$, and so on.)

In any sequence satisfying Definition 14, there will be a fixed-point. Theorems 2 and 3 are preparatory for this result, stated in Theorem 4.

Theorem 2 *For any $\mathcal{L}[\mathrm{T,G}]$ sentence p and any $\mathcal{L}[\mathrm{T,G}]$ models $\mathcal{M}[\mathrm{T,G}]$ and $\mathcal{M}[\mathrm{T',G'}]$, where $\mathcal{M}[\mathrm{T',G'}]$ extends $\mathcal{M}[\mathrm{T,G}]$:*
if $v(\mathcal{M}[\mathrm{T,G}])(p) \in \{0,1\}$, $v(\mathcal{M}[\mathrm{T',G'}])(p) = v(\mathcal{M}[\mathrm{T,G}])(p)$.

This theorem establishes that, if an $\mathcal{L}[\mathrm{T,G}]$ sentence receives a classical value (0 or 1) in a model $\mathcal{M}[\mathrm{T,G}]$, it receives the same classical value in any model extending $\mathcal{M}[\mathrm{T,G}]$. The proof of Theorem 2 is a simple induction on formula complexity.

Theorem 3 (Monotonicity Conditions for † Construction) *For all ordinals α and β:*

- *If $\alpha > \beta$, then $\mathcal{M}[\mathrm{T,G}]_\alpha$ extends $\mathcal{M}[\mathrm{T,G}]_\beta$.*
- *If $\alpha > \beta$, then for all $\mathcal{L}[\mathrm{T,G}]$ sentences p, if $v(\mathcal{M}[\mathrm{T,G}]_\beta)(p) \in \{0,1\}$, then $v(\mathcal{M}[\mathrm{T,G}]_\alpha)(p) = v(\mathcal{M}[\mathrm{T,G}]_\beta)(p)$.*

Proof: The first Monotonicity Condition implies the second, given Theorem 2, so it is sufficient to prove the first. If α is 0 or a limit ordinal the condition holds trivially. So it suffices to show that the first condition holds for an ordinal $\alpha + 1$ on the assumption that both Monotonicity Conditions hold for any ordinal $\leq \alpha$. We do this by showing that $\mathrm{T}_+^\alpha \subseteq \mathrm{T}_+^{\alpha+1}$, $\mathrm{T}_-^\alpha \subseteq \mathrm{T}_-^{\alpha+1}$, $\mathrm{G}_+^\alpha \subseteq \mathrm{G}_+^{\alpha+1}$, and $\mathrm{G}_-^\alpha \subseteq \mathrm{G}_-^{\alpha+1}$.

- Suppose that o is an element of T_+^α. Let γ be the least ordinal such that $o \in \mathrm{T}_+^\gamma$. γ cannot be a limit ordinal, so γ is either 0 or a successor. Suppose $\gamma = 0$. By the assumption that $\mathcal{M}[\mathrm{T,G}]$ is in good standing, it follows that o is an $\mathcal{L}[\mathrm{T,G}]$ sentence and $v(\mathcal{M}[\mathrm{T,G}]_\gamma)(o) = 1$. Since we assume that both Monotonicity conditions hold for all ordinals $\leq \alpha$, and $\gamma \leq \alpha$, it follows that $v(\mathcal{M}[\mathrm{T,G}]_\alpha)(o) = 1$. Therefore, by definition of the sequence, $o \in \mathrm{T}_+^{\alpha+1}$. Suppose that γ is a successor. Then there is some ordinal that is $\gamma - 1$. By definition of the sequence, $v(\mathcal{M}[\mathrm{T,G}]_{\gamma-1})(o) = 1$. Since we assume that both Monotonicity conditions hold for all ordinals $\leq \alpha$, and $\gamma - 1 < \alpha$, it follows that $v(\mathcal{M}[\mathrm{T,G}]_\alpha)(o) = 1$. Therefore, by definition of the sequence $o \in \mathrm{T}_+^{\alpha+1}$.

- Suppose o is an element of T_-^α. Either o is a sentence of $\mathcal{L}[\mathrm{T,G}]$ or it is not. If it is not, then, by Definition 14 and the assumption that $\mathcal{M}[\mathrm{T,G}]$ is in good standing, for every ordinal γ, $o \in \mathrm{T}_-^\gamma$. So, $o \in \mathrm{T}_-^{\alpha+1}$, if o is not a sentence of $\mathcal{L}[\mathrm{T,G}]$. If o is an $\mathcal{L}[\mathrm{T,G}]$ sentence, then there is a proof that $o \in \mathrm{T}_-^{\alpha+1}$ that exactly parallels the proof above that $o \in \mathrm{T}_+^{\alpha+1}$.

- The proof for $\mathrm{G}_{+}^{\alpha+1}$ has the same form as the proof for $\mathrm{T}_{+}^{\alpha+1}$. The proof of $\mathrm{G}_{-}^{\alpha+1}$ is simpler because, by Definition 10, $J^{\dagger}$ always preserves the contents of G_{-}. Therefore $\mathrm{T}_{+}^{\alpha} \subseteq \mathrm{T}_{+}^{\alpha+1}$, $\mathrm{T}_{-}^{\alpha} \subseteq \mathrm{T}_{-}^{\alpha+1}$, $\mathrm{G}_{+}^{\alpha} \subseteq \mathrm{G}_{+}^{\alpha+1}$, and $\mathrm{G}_{-}^{\alpha} \subseteq \mathrm{G}_{+}^{\alpha+1}$.

So both Monotonicity Conditions hold for all ordinals α and β. QED

Theorem 4 († Fixed-Point) *There is an ordinal β such that $\mathcal{M}[\mathrm{T,G}]_{\beta} = \mathcal{M}[\mathrm{T,G}]_{\beta+1}$.*

This follows since there are ordinals that exceed the cardinality of any set of $\mathcal{L}[\mathrm{T,G}]$ sentences. It follows further that, if $\mathcal{M}[\mathrm{T,G}]_{\beta}$ is a fixed-point in some † construction, then for any $\alpha > \beta$, $\mathcal{M}[\mathrm{T,G}]_{\beta} = \mathcal{M}[\mathrm{T,G}]_{\alpha}$. So for a given † construction beginning with a model $\mathcal{M}$ we can speak of *the* fixed-point for the † construction beginning with $\mathcal{M}$.

For the sake of completeness, I'll also define the standard Kripke construction (the $*$ Construction) and state the analogues of Theorems 2-4, with proofs left to the reader.

Definition 15 *Suppose that $\mathcal{M}[\mathrm{T}]$ is an $\mathcal{L}[\mathrm{T}]$ model that is in good standing. We define a sequence of $\mathcal{L}[\mathrm{T}]$ models as follows:*

(i) $\mathcal{M}[\mathrm{T}]_0 = \mathcal{M}[\mathrm{T}]$
(ii) $\mathcal{M}[\mathrm{T}]_{\alpha+1} = J^{*}(\mathcal{M}[\mathrm{T}]_{\alpha})$
(iii) Where α is a limit ordinal, $\mathcal{M}[\mathrm{T}]_{\alpha} = \mathcal{M}[\langle \bigcup_{\beta<\alpha} \mathrm{T}_{+}^{\beta}, \bigcup_{\beta<\alpha} \mathrm{T}_{-}^{\beta} \rangle]$

Theorem 5 *For any $\mathcal{L}[\mathrm{T}]$ sentence p and any $\mathcal{L}[\mathrm{T}]$ models $\mathcal{M}[\mathrm{T}]$ and $\mathcal{M}[\mathrm{T}']$, where $\mathcal{M}[\mathrm{T}']$ extends $\mathcal{M}[\mathrm{T}]$:*

if $v(\mathcal{M}[\mathrm{T}])(p) \in \{0,1\}$, $v(\mathcal{M}[\mathrm{T}'])(p) = v(\mathcal{M}[\mathrm{T}])(p)$.

Theorem 6 (Monotonicity Conditions for $\star$ Construction) *For all ordinals α and β:*

- *If $\alpha > \beta$, then $\mathcal{M}_{\alpha}^{*}$ extends $\mathcal{M}_{\beta}^{*}$.*
- *If $\alpha > \beta$, then for all $\mathcal{L}[\mathrm{T}]$ sentences p, if $v(\mathcal{M}_{\beta}^{*})(p) \in \{0,1\}$, then $v(\mathcal{M}_{\alpha}^{*})(p) = v(\mathcal{M}_{\beta}^{*})(p)$.*

Theorem 7 ($*$ Fixed-Point) *There is an ordinal β such that $\mathcal{M}_{\beta}^{*} = \mathcal{M}_{\beta+1}^{*}$.*

3 Comparison of * and † Constructions

Suppose that we have an $\mathcal{L}[\mathrm{T}]$ model $\mathcal{M}_0^{*}$ and an $\mathcal{L}[\mathrm{T,G}]$ model $\mathcal{M}_0^{\dagger}$ that corresponds to $\mathcal{M}_0^{*}$, and that both are in good standing. Since $\mathcal{M}_0^{\dagger}$ and $\mathcal{M}_0^{*}$ correspond, the objects in their respective domains that are not sentences of $\mathcal{L}[\mathrm{T,G}]$ are exactly the same. For the rest of this section, I will call the set of such objects Q. Let $\mathcal{M}^{*}$ refer to the fixed-point for the $*$ construction beginning with $\mathcal{M}_0^*$ as a base model, and let $\mathcal{M}^\dagger$ refer to the fixed-point for the $\dagger$ construction beginning with $\mathcal{M}_0^\dagger$ as a base model. This section proves a number of notable results relating $\mathcal{M}^*$ and $\mathcal{M}^\dagger$, substantiating my suggestion that, in $\mathcal{L}$[T,G] we can express Kripke's diagnostic insight.

I'll briefly summarize the organization of what follows. Lemmas 8-11 prepare the way for the first important point of correspondence between $\mathcal{M}^*$ and $\mathcal{M}^\dagger$, stated in Theorem 12 and Corollary 12.1, namely: that the truth-concepts of $\mathcal{M}^*$ and $\mathcal{M}^\dagger$ agree on every sentence of $\mathcal{L}$[T] and every $\mathcal{L}$[T] sentence receives the same value in $\mathcal{M}^*$ and $\mathcal{M}^\dagger$. Everything up to Corollary 12.1 concerns only the handling of $\mathcal{L}$[T] sentences in the respective models—the results do not turn on the interpretation of G.

Lemmas 13 and 14 are preliminaries for the main upshots concerning the G predicate: Theorems 15 and 16. Theorem 15 states that, once a sentence is added to the anti-extension G, it never receives a classical value at any further stage in the $\dagger$ construction. This is presupposed by Theorem 16, the main result of the paper. It states that, for any $\mathcal{L}$[T] sentence p, $v(\mathcal{M}^*)(p) = \frac{1}{2}$ iff p is a member of $\mathrm{G}_-^\dagger$, i.e. the G_- of $\mathcal{M}^\dagger$. In other words, among $\mathcal{L}$[T] sentences, $\mathrm{G}_-^\dagger$ includes all and only the sentences that are ungrounded in $\mathcal{M}^*$.

Lemma 8 *For any $\mathcal{L}$[T] model $\mathcal{M}$, any $\mathcal{L}$[T,G] model $\mathcal{M}'$, and any $\mathcal{L}$[T] sentence p, if $\mathcal{M}$ and $\mathcal{M}'$ correspond, then $v(\mathcal{M})(p) = v(\mathcal{M}')(p)$.*

The proof of this is a simple induction on formula complexity for $\mathcal{L}$[T] sentences. If an $\mathcal{L}[T]$ and $\mathcal{L}$[T,G] correspond, then they are identical in all respects that are relevant to assessing $\mathcal{L}$[T].

Lemma 9 *For any sentence p of $\mathcal{L}$ and any ordinal α, $v(\mathcal{M}_\alpha^*)(p) = v(\mathcal{M}_\alpha^\dagger)(p)$*

$\mathcal{L}$ sentences do not feature occurrences of T or G, so their valuation is invariant according to the definitions of the respective constructions. The proof of this lemma is a transfinite induction that I likewise leave to the reader.

Since $\mathcal{M}_0^*$ and $\mathcal{M}_0^\dagger$ correspond, for any term c in $\mathcal{L}$[T], $\delta(\mathcal{M}_0^*)(c) = \delta(\mathcal{M}_0^\dagger)(c)$. And since the definitions of the respective constructions preserve the denotation function, for any ordinals α and β, $\delta(\mathcal{M}_\alpha^*)(c) = \delta(\mathcal{M}_\beta^*)(c)$ and $\delta(\mathcal{M}_\alpha^\dagger)(c) = \delta(\mathcal{M}_\beta^\dagger)(c)$. So for any term c in $\mathcal{L}$[T] we can speak univocally of *the denotation* of c throughout both of the constructions under consideration.

Lemma 10 *For any ordinal α and any $\mathcal{L}[T]$ sentence p, if p is $\mathrm{T}(c)$, where the denotation of c is in Q, then $v(\mathcal{M}_\alpha^*)(p) = v(\mathcal{M}_\alpha^\dagger)(p) = 0$.*

The proof of this is a simple transfinite induction. By the assumption that $\mathcal{M}^*_0$ and $\mathcal{M}^\dagger_0$ are both in good standing, and the assumption that the denotation of $c \in Q$, the denotation of $c \in \mathrm{T}^{*0}_- \cup \mathrm{T}^{\dagger 0}_-$. So $v(\mathcal{M}^*_\alpha)(\mathrm{T}(c)) = v(\mathcal{M}^\dagger_\alpha)(\mathrm{T}(c)) = 0$. The definition of the respective constructions guarantees that every member of Q remains in the anti-extension of T.

Lemma 11 *For any ordinal α and any $\mathcal{L}[\mathrm{T}]$ sentence p, if $p \in \mathrm{T}^{*\alpha}_+$ iff $p \in \mathrm{T}^{\dagger\alpha}_+$ and $p \in \mathrm{T}^{*\alpha}_-$ iff $p \in \mathrm{T}^{\dagger\alpha}_-$, then $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$.*

Proof: Let α be an arbitrary ordinal, and suppose that for any $\mathcal{L}[\mathrm{T}]$ sentence r, $r \in \mathrm{T}^{*\alpha}_+$ iff $r \in \mathrm{T}^{\dagger\alpha}_+$ and $r \in \mathrm{T}^{*\alpha}_-$ iff $r \in \mathrm{T}^{\dagger\alpha}_-$. Let p be an arbitrary sentence of $\mathcal{L}[\mathrm{T}]$. Now we show by induction on formula complexity that $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$.

- Suppose p is a non-alethic atomic. Then p is a sentence of $\mathcal{L}$, and, by Lemma 9, $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$.
- Suppose p is $\mathrm{T}(c)$. Either the denotation of $c \in Q$ or the denotation of $c \notin Q$. If the denotation of $c \in Q$, then, by Lemma 10, it follows that $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$. Suppose that the denotation of $c \notin Q$. Let s be the $\mathcal{L}[\mathrm{T}]$ sentence denoted by c in $\mathcal{M}^*_\alpha$ and $\mathcal{M}^\dagger_\alpha$. Suppose for contradiction that $v(\mathcal{M}^*_\alpha)(\mathrm{T}(c)) \neq v(\mathcal{M}^\dagger_\alpha)(\mathrm{T}(c))$. In that case, it must be that $s \in \mathrm{T}^{*\alpha}_+ \cup \mathrm{T}^{\dagger\alpha}_+ \cup \mathrm{T}^{*\alpha}_- \cup \mathrm{T}^{\dagger\alpha}_-$, or otherwise $\mathrm{T}(c)$, would receive $\frac{1}{2}$ in both $\mathcal{M}^*_\alpha$ and $\mathcal{M}^\dagger_\alpha$. Since, $s \in \mathrm{T}^{*\alpha}_+$ iff $s \in \mathrm{T}^{\dagger\alpha}_+$ and $s \in \mathrm{T}^{*\alpha}_-$ iff $s \in \mathrm{T}^{\dagger\alpha}_-$, either $s \in \mathrm{T}^{*\alpha}_+ \cup \mathrm{T}^{\dagger\alpha}_+$ or $s \in \mathrm{T}^{*\alpha}_- \cup \mathrm{T}^{\dagger\alpha}_-$. By the semantics in Definitions 2 and 4, either $v(\mathcal{M}^*_\alpha)(\mathrm{T}(c)) = v(\mathcal{M}^\dagger_\alpha)(\mathrm{T}(c)) = 1$ or $v(\mathcal{M}^*_\alpha)(\mathrm{T}(c)) = v(\mathcal{M}^\dagger_\alpha)(\mathrm{T}(c)) = 0$. So in either case, $v(\mathcal{M}^*_\alpha)(\mathrm{T}(c)) = v(\mathcal{M}^\dagger_\alpha)(\mathrm{T}(c))$. This contradicts our assumption for *reductio*. Therefore, if p is $\mathrm{T}(c)$, $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$.
- Suppose p is $\neg\phi$, and assume the inductive hypothesis holds for ϕ: $v(\mathcal{M}^*_\alpha)(\phi) = v(\mathcal{M}^\dagger_\alpha)(\phi)$. Then by the semantics $v(\mathcal{M}^*_\alpha)(\neg\phi) = v(\mathcal{M}^\dagger_\alpha)(\neg\phi) = 1 - v(\mathcal{M}^*_\alpha)(\phi)$. So if p is $\neg\phi$, $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$.
- The proofs for the cases where p is $\phi \vee \psi$ or $\exists v\phi$ follow the same pattern as the proof for $\neg\phi$.

Therefore, $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$. Since α and p were arbitrary, it follows that, for any ordinal α and any $\mathcal{L}[\mathrm{T}]$ sentence p, if $p \in \mathrm{T}^{*\alpha}_+$ iff $p \in \mathrm{T}^{\dagger\alpha}_+$ and $p \in \mathrm{T}^{*\alpha}_-$ iff $p \in \mathrm{T}^{\dagger\alpha}_-$, then $v(\mathcal{M}^*_\alpha)(p) = v(\mathcal{M}^\dagger_\alpha)(p)$. QED

Theorem 12 *For any ordinal α and any $\mathcal{L}[\mathrm{T}]$ sentence p, $p \in \mathrm{T}^{*\alpha}_+$ iff $p \in \mathrm{T}^{\dagger\alpha}_+$ and $p \in \mathrm{T}^{*\alpha}_-$ iff $p \in \mathrm{T}^{\dagger\alpha}_-$.*

Proof: Proof by transfinite induction. For $\alpha = 0$, the condition is trivial on the assumption that $\mathcal{M}^*_0$ and $\mathcal{M}^\dagger_0$ correspond. For $\alpha = \beta + 1$, suppose that for any $\mathcal{L}$[T] sentence p, $p \in \mathrm{T}^{*\beta}_+$ iff $p \in \mathrm{T}^{\dagger\beta}_+$ and $p \in \mathrm{T}^{*\beta}_-$ iff $p \in \mathrm{T}^{\dagger\beta}_-$. Then, by Lemma 11, for every sentence p of $\mathcal{L}$[T], $v(\mathcal{M}^*_\beta)(p) = v(\mathcal{M}^\dagger_\beta)(p)$. Therefore, by the definitions of the respective constructions, for every sentence p of $\mathcal{L}$[T], $p \in \mathrm{T}^{*\alpha}_+$ iff $p \in \mathrm{T}^{\dagger\alpha}_+$ and $p \in \mathrm{T}^{*\alpha}_-$ iff $p \in \mathrm{T}^{\dagger\alpha}_-$. Now we prove that the condition holds for a limit ordinal α, on the assumption that it holds for all $\beta < \alpha$. According to definitions of the respective constructions, $\mathrm{T}^{\dagger\alpha}_+ = \bigcup_{\beta<\alpha} \mathrm{T}^{\dagger\beta}_+$, $\mathrm{T}^{\dagger\alpha}_- = \bigcup_{\beta<\alpha} \mathrm{T}^{\dagger\beta}_-$, $\mathrm{T}^{*\alpha}_+ = \bigcup_{\beta<\alpha} \mathrm{T}^{*\beta}_+$, and $\mathrm{T}^{*\alpha}_- = \bigcup_{\beta<\alpha} \mathrm{T}^{*\beta}_-$. Since we are assuming that, for all $\beta < \alpha$, $p \in \mathrm{T}^{*\beta}_+$ iff $p \in \mathrm{T}^{\dagger\beta}_+$ and $p \in \mathrm{T}^{*\beta}_-$ iff $p \in \mathrm{T}^{\dagger\beta}_-$, it follows that $p \in \mathrm{T}^{*\alpha}_+$ iff $p \in \mathrm{T}^{\dagger\alpha}_+$ and $p \in \mathrm{T}^{*\alpha}_-$ iff $p \in \mathrm{T}^{\dagger\alpha}_-$. QED

Corollary 12.1 *For any $\mathcal{L}$[T] sentence p, $v(\mathcal{M}^\dagger)(p) = v(\mathcal{M}^*)(p)$.*

This follows from Lemmas 11 and 12. Every $\mathcal{L}$[T] sentence has the same value in $\mathcal{M}^\dagger$ and $\mathcal{M}^*$.

Lemma 13 is a helpful preliminary for the proof of Lemma 14. In order to state these lemmas it will be useful to introduce some additional terminology regarding alethic sentences.

Definition 16 *If p is an $\mathcal{L}$[T,G] sentence, then q is an **anchor** of p iff one of the following holds:*

(i) p contains an occurrence of $T(<\phi>)$ and q is ϕ
(ii) p contains an occurrence of $T(v)$, where v is a variable, and q is any $\mathcal{L}$[T,G] sentence

A sentence x is an anchor of a sentence y if and only if there is a specific sort of calling path linking x and y: namely a calling path that goes through one direct-call occasioned by an occurrence of T. The set of all p's anchors is the **sufficiency set** of p.

Lemma 13 *If p is an $\mathcal{L}$[T] sentence with a non-empty sufficiency set H, and p receives a classical value (0 or 1) on a model $\mathcal{M}$[T,G], then, for any model $\mathcal{M}[\mathrm{T}', \mathrm{G}']$ if T'_+ is such that $(H \cap \mathrm{T}_+) \subseteq \mathrm{T}'_+$, and T'_- is such that $(H \cap \mathrm{T}_-) \subseteq \mathrm{T}'_-$, then $v(\mathcal{M}[\mathrm{T}, \mathrm{G}])(p) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(p)$.*

Proof: Suppose that p is an $\mathcal{L}$[T] sentence with a non-empty sufficiency set H, p receives a classical value on $\mathcal{M}$[T,G], and $\mathcal{M}[\mathrm{T}', \mathrm{G}']$ is a model such that$(H \cap \mathrm{T}_+) \subseteq \mathrm{T}'_+$ and $(H \cap \mathrm{T}_-) \subseteq \mathrm{T}'_-$. We prove by induction on formula complexity that $v(\mathcal{M}[\mathrm{T,G}])(p) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(p)$:

- Suppose p is a non-alethic atomic. Then the condition holds trivially because, if p is non-alethic, its interpretation is invariant with respect to changes in the interpretation of T.

- Suppose p is $\mathrm{T}(c)$. Either the denotation of $c \in Q$ or $c \notin Q$. Suppose the former. Then, since c does not denote an $L[\mathrm{T}]$ sentence, the sufficiency set of $\mathrm{T}(c)$ is $\varnothing$. This contradicts the assumption that H is non-empty.

  If the denotation of $c \notin Q$, then c denotes some $\mathcal{L}[\mathrm{T,G}]$ sentence—call it s. Then $H = \{s\}$. We show now that, whichever classical value $\mathrm{T}(c)$ receives on $\mathcal{M}[\mathrm{T,G}]$, $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T}(c)) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\mathrm{T}(c))$. Suppose, $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T}(c)) = 1$. By Definition 4, $s \in T_+$, so $H \cap \mathrm{T}_+ = \{s\}$. It follows from our assumption above that $\{s\} \subseteq \mathrm{T}'_+$. Therefore, by Definition 4, $v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\mathrm{T}(c)) = 1$. So if $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T(c)}) = 1$, $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T}(c)) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\mathrm{T}(c))$. Suppose $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T}(c)) = 0$. By Definition 4, $s \in \mathrm{T}_-$, so $H \cap \mathrm{T}_- = \{s\}$. It follows from our assumption above that $\{s\} \subseteq \mathrm{T}'_-$. Therefore, by Definition 4, $v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\mathrm{T}(c)) = 0$. So if $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T(c)}) = 0$, $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T}(c)) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\mathrm{T}(c))$. Therefore, whichever classical value $\mathrm{T}(c)$ receives on $\mathcal{M}[\mathrm{T,G}]$, $v(\mathcal{M}[\mathrm{T,G}])(\mathrm{T}(c)) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\mathrm{T}(c))$. So the condition holds if p is $\mathrm{T}(c)$.

- Suppose p is $\neg\phi$. We assume the inductive hypothesis holds for ϕ: $v(\mathcal{M}[\mathrm{T,G}])(\phi) = v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\phi)$. By assumption, $\neg\phi$ receives a classical value on $\mathcal{M}[\mathrm{T,G}]$. Let that value be x. Then, by the semantics in Definition 4, ϕ receives the value $|x - 1|$ on $\mathcal{M}[\mathrm{T,G}]$. By inductive hypothesis, $v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\phi) = |x - 1|$. So, by the semantics in Definition 4, $v(\mathcal{M}[\mathrm{T}', \mathrm{G}'])(\neg\phi) = x$. So the condition holds if p is $\neg\phi$.

- The proofs for $\ulcorner \phi \vee \psi \urcorner$ and $\ulcorner \exists v \phi \urcorner$ follow the same pattern as $\neg\phi$.

- QED

This lemma holds for all $\mathcal{L}[\mathrm{T}]$ models as well, since it only concerns $\mathcal{L}[\mathrm{T}]$ sentences. In a slogan this lemma is: for $\mathcal{L}[\mathrm{T}]$ sentences, *only anchors matter.*

The next lemma proves a point that is important for the proof of Theorem 15. That Theorem shows that, once a sentence is added to G_- in the † construction, it never receives a classical value at any subsequent stage.

Lemma 14 *For any ordinal β, if Z is an ungrounded set on $\mathcal{M}^{\dagger}_{\beta}$, then, for any ordinal $\alpha > \beta$, if no member of Z is a member of $\mathrm{T}^{\dagger\alpha}_{+}$ or $\mathrm{T}^{\dagger\alpha}_{-}$, no member of Z receives a classical value in $\mathcal{M}^{\dagger}_{\alpha}$.*

Proof: Suppose Z is an ungrounded set on $\mathcal{M}^{\dagger}_{\beta}$. Let γ be an arbitrary ordinal such that $\gamma > \beta$. Suppose that no member of Z is a member of $\mathrm{T}^{\dagger\gamma}_{+}$ or $\mathrm{T}^{\dagger\gamma}_{-}$. Suppose now for contradiction that some $o \in Z$ has a classical value on $\mathcal{M}^{\dagger}_{\gamma}$. Let $F = \mathrm{T}^{\dagger\gamma}_{+} \cap \{x \mid x \text{ is an anchor of } o\}$ and $E = \mathrm{T}^{\dagger\gamma}_{-} \cap \{x \mid x \text{ is an anchor of } o\}$. $F \cup E$ is disjoint from Z. Since Z is an ungrounded set in $\mathcal{M}^{\dagger}_{\beta}$ and is therefore closed under direct call among sentences that receive $\frac{1}{2}$ in $\mathcal{M}^{\dagger}_{\beta}$, it follows that every member of F or E receives a classical value in $\mathcal{M}^{\dagger}_{\beta}$. This follows because $o \in Z$ and, by the definition of anchoring, there is a direct calling path from o to every member of $F \cup E$. Since Z and $F \cup E$ are disjoint, it must be that no member of $F \cup E$ receives $\frac{1}{2}$ in $\mathcal{M}^{\dagger}_{\beta}$.

We now show that $F \subseteq \mathrm{T}^{\dagger\beta+1}_{+}$ and $E \subseteq \mathrm{T}^{\dagger\beta+1}_{-}$. Let h be an arbitrary member of F. Since h has a classical value in $\mathcal{M}^{\dagger}_{\beta}$, it follows from Definition 14, that h is either a member of $\mathrm{T}^{\dagger\beta+1}_{+}$ or $\mathrm{T}^{\dagger\beta+1}_{-}$. Since the extension and anti-extension of T are disjoint in every $\mathcal{L}$[T,G] model and $h \in \mathrm{T}^{\dagger\gamma}_{+}$, it follows $h \notin \mathrm{T}^{\dagger\gamma}_{-}$. Since $\gamma \geq \beta + 1$, the first Monotonicity Condition of Theorem 3 implies that $h \notin \mathrm{T}^{\dagger\beta+1}_{-}$. Therefore, $h \in \mathrm{T}^{\dagger\beta+1}_{+}$. Since h was arbitrary, we can conclude that $F \subseteq \mathrm{T}^{\dagger\beta+1}_{+}$. A parallel argument shows that $E \subseteq \mathrm{T}^{\dagger\beta+1}_{-}$.

Since $F \subseteq \mathrm{T}^{\dagger\beta+1}_{+}$ and $E \subseteq \mathrm{T}^{\dagger\beta+1}_{-}$, then, by Lemma 13, o receives a classical value in $\mathcal{M}^{\dagger}_{\beta+1}$, because $F = \mathrm{T}^{\dagger\gamma}_{+} \cap \{x \mid x \text{ is an anchor of } o\}$ and $E = \mathrm{T}^{\dagger\gamma}_{-} \cap \{x \mid x \text{ is an anchor of } o\}$ and o receives a classical value in $\mathcal{M}^{\dagger}_{\gamma}$. We now do an induction on formula complexity to show that $\mathcal{M}^{\dagger}_{\beta}$ settles o.

- If o is a non-alethic atomic or atomic of the form G(c) the condition holds trivially. Such sentences do not directly call anything, so it follows that, in the dependence graph $< R, Z >$, o is a terminal node. This contradicts the assumption that Z is an ungrounded set in $\mathcal{M}^{\dagger}_{\beta}$. Therefore, if o is a non-alethic atomic or atomic of the form G(c), $\mathcal{M}^{\dagger}_{\beta}$ settles o.

- Suppose o is T(c). Given that Z is an ungrounded set, T(c) directly calls something, so c denotes an $\mathcal{L}$[T,G] sentence. Call that sentence s. Since T(c) has a classical value in $\mathcal{M}^{\dagger}_{\beta+1}$, the definition of the † construction implies that s has a classical value in $\mathcal{M}^{\dagger}_{\beta}$. It follows that $\mathcal{M}^{\dagger}_{\beta}$ settles T(c). So the condition holds if o is T(c).

- Suppose o is $\neg\phi$. Assume the inductive hypothesis that ϕ is settled on $\mathcal{M}^{\dagger}_{\beta}$. Then $\neg\phi$ is settled on $\mathcal{M}^{\dagger}_{\beta}$. So o is settled on $\mathcal{M}^{\dagger}_{\beta}$ in the case where o is $\neg\phi$. Parallel arguments show that o is settled on $\mathcal{M}^{\dagger}_{\beta}$ if o is $\phi \vee \psi$ or $\exists v\phi$.

Therefore o is settled on $\mathcal{M}^{\dagger}_{\beta}$. But this contradicts the assumption that Z is an ungrounded set on $\mathcal{M}^{\dagger}_{\beta}$, since $o \in Z$ and no member of an ungrounded set relative to a model is settled by that model. Discharging our assumption for contradiction: no member of Z receives a classical value in $\mathcal{M}^{\dagger}_{\gamma}$. Since γ was arbitrary, it follows that for any ordinal $\alpha > \beta$, if no member of Z is a member of $\mathrm{T}^{\dagger\alpha}_{+}$ or $\mathrm{T}'^{\dagger\alpha}_{-}$, no member of Z receives a classical value in $\mathcal{M}^{\dagger}_{\alpha}$. QED

Informally, if Z is an ungrounded set relative to $\mathcal{M}^{\dagger}_{\beta}$, then, for any ordinal $\alpha > \beta$, it is a necessary condition for any member of Z to receive a classical value in $\mathcal{M}^{\dagger}_{\alpha}$ that some member of Z is added to the truth-concept of $\mathcal{M}^{\dagger}_{\alpha}$.

Theorem 15 is necessary for establishing the main result of this paper: that for all $\mathcal{L}[T]$ sentences p, $v(\mathcal{M}^*)(p) = \frac{1}{2}$ iff p is a member of $\mathrm{G}^{\dagger}_{-}$ (Theorem 16). But it is also of independent interest in characterizing constructions that lead to $\mathcal{L}$[T,G] fixed-point models.

Theorem 15 *For any ordinal τ and any $\mathcal{L}$[T,G] sentence p, if p is a member of G^{τ}_{-}, then, for all $\beta \geq \tau$, $v(\mathcal{M}^{\dagger}_{\beta})(p) = \frac{1}{2}$.*

Proof: Let p be an arbitrary sentence of $\mathcal{L}$[T,G]. Let α be the least ordinal such that $p \in G^{\dagger\alpha}_{-}$. α cannot be 0, since $\mathcal{M}^{\dagger}_{0}$ is stipulated to be in good standing. Nor can α be a limit ordinal, since, by Definition 14, the definition of the † construction, for every limit ordinal κ, every member of $\mathrm{G}^{\dagger}_{-}$ is a member of $\mathrm{G}^{\dagger\chi}_{-}$ for some $\chi < \kappa$. So α is a successor ordinal. By the definition of the † construction, p is a member of an ungrounded set in $\mathcal{M}^{\dagger}_{\alpha-1}$. Call that set U. We now show by transfinite induction that for every ordinal $\beta \geq \alpha - 1$, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\beta})(o) = \frac{1}{2}$.

For the case where $\beta = \alpha - 1$, let o be an arbitrary member of U. Given that U is an ungrounded set in $\mathcal{M}^{\dagger}_{\alpha-1}$, it follows that o is not settled by $\mathcal{M}^{\dagger}_{\alpha-1}$. Therefore, $v(\mathcal{M}^{\dagger}_{\alpha-1})(o) = \frac{1}{2}$. Since o was arbitrary, it follows that, in the case where $\beta = \alpha - 1$, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\beta})(o) = \frac{1}{2}$. For the case where $\beta = \gamma + 1$ for some $\gamma \geq \alpha - 1$, we assume the inductive hypothesis that, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\gamma})(o) = \frac{1}{2}$. Suppose for *reductio*, that some sentence $s \in U$ has value 1 or 0 on $\mathcal{M}^{\dagger}_{\beta}$. Since U is an ungrounded set in $\mathcal{M}^{\dagger}_{\alpha-1}$ and $\beta > \alpha - 1$, Lemma 14 implies that if no member of U is a member of $\mathrm{T}^{\dagger\beta}_{+}$ or $\mathrm{T}^{\dagger\beta}_{-}$, no member of U receives a classical value in $\mathcal{M}^{\dagger}_{\beta}$. So, on the assumption s receives a classical value on $\mathcal{M}^{\dagger}_{\beta}$, it follows that some member of U is a member of T^{β}_{+} or T^{β}_{-}. But, given Definition 14, the definition of the † construction, this implies that some member of U has a classical value in $\mathcal{M}^{\dagger}_{\gamma}$—contrary to hypothesis. Therefore, in the case where $\beta = \gamma + 1$, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\beta})(o) = \frac{1}{2}$. For the case where β is a limit ordinal greater than $\alpha - 1$, suppose that for all $\gamma < \beta$, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\gamma})(o) = \frac{1}{2}$. The definition of the † construction implies that nothing receives a classical value at a limit ordinal that does not receive it at some predecessor ordinal. Therefore, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\beta})(o) = \frac{1}{2}$. This completes the transfinite induction. So we can conclude for any $\beta \geq \alpha - 1$, for all $o \in U$, $v(\mathcal{M}^{\dagger}_{\beta})(o) = \frac{1}{2}$. Since $p \in U$, for all $\beta \geq \alpha - 1$, $v(\mathcal{M}^{\dagger}_{\beta})(p) = \frac{1}{2}$. Since α is the least ordinal such that $p \in \mathrm{G}^{\dagger\alpha}_{-}$, it follows, for any ordinal τ, if p is a member of G^{τ}_{-}, then, for all $\beta \geq \tau$, $v(\mathcal{M}^{\dagger}_{\beta})(p) = \frac{1}{2}$. Since p was arbitrary this holds for all sentences of $\mathcal{L}$[T,G]. QED

Now we prove the main result regarding the correspondence between $\mathcal{M}^*$ and $\mathcal{M}^{\dagger}$.

Theorem 16 *For all $\mathcal{L}$[T] sentences p, $v(\mathcal{M}^*)(p) = \frac{1}{2}$ iff p is a member of $\mathrm{G}^{\dagger}_{-}$, i.e. the G_{-} of $\mathcal{M}^{\dagger}$.*

Proof:

Left to Right: For all $\mathcal{L}[\mathrm{T}]$ sentences p, if $v(\mathcal{M}^*)(p) = \frac{1}{2}$, then p is a member of $\mathrm{G}^\dagger_-$. Induction on formula complexity.

- Suppose p is a non-alethic atomic. Trivial, since p has a classical value in $\mathcal{M}^*$.

- Suppose p is $\mathrm{T}(c)$. Suppose that $v(\mathcal{M}^*)(\mathrm{T}(c)) = \frac{1}{2}$. There is no ordinal α such that $\mathcal{M}^\dagger_\alpha$ gives $\mathrm{T}(c)$ a classical value, because if it did $\mathrm{T}(c)$ would have a classical value on $\mathcal{M}^\dagger$, and, by Corollary 12.1, $\mathrm{T}(c)$ would have a classical value on $\mathcal{M}^*$ contrary to assumption. It follows too that there is no ordinal α such that $\mathcal{M}^\dagger_\alpha$ settles $\mathrm{T}(c)$. Let H be the smallest set including $\mathrm{T}(c)$ such that $\forall x \in H : \forall y(Rxy \to y \in H)$, i.e. the smallest set containing T(c) that is closed under direct call. Let β be the least ordinal such that every member of H that receives a classical value in $\mathcal{M}^\dagger$ receives a classical value in $\mathcal{M}^\dagger_\beta$. Let $H' = H - \{x \mid x \text{ receives a classical value in } \mathcal{M}^\dagger_\beta\}$. We can prove that H' is an ungrounded set in $\mathcal{M}^\dagger_\beta$.

  By hypothesis, no member of H' receives a classical value in $\mathcal{M}^\dagger_\beta$ and it is closed under direct call among sentences that receive $\frac{1}{2}$ in $\mathcal{M}^\dagger_\beta$. Every member $o \in H'$ must be unsettled on $\mathcal{M}^\dagger_\beta$, because if o were settled, then o would receive a classical value in $J^\dagger(\mathcal{M}^\dagger_\beta)$, and therefore in $\mathcal{M}^\dagger$, contrary to hypothesis. All that remains is to show that the dependence graph $< R, H' >$ has no terminal nodes. To do this we show, for an arbitrary $o \in H'$ that it directly calls some member of H'. o is an alethic sentence, because every non-alethic sentence of $\mathcal{L}[\mathrm{T}]$ has a classical value in $\mathcal{M}^\dagger_\beta$. Suppose for contradiction that, for every sentence q such that o directly calls q, q has a classical value in $\mathcal{M}^\dagger_\beta$. In that case, Definitions 5, 7, 8, imply that o is settled by $\mathcal{M}^\dagger_\beta$. This contradicts the assertion above that every member of H' is unsettled on $\mathcal{M}^\dagger_\beta$. Therefore, for all sentences q, if o directly calls q, then $v(\mathcal{M}^\dagger_\beta)(q) = \frac{1}{2}$. Since $o \in H'$, H' is closed under direct call among sentences that receive $\frac{1}{2}$ in $\mathcal{M}^\dagger_\beta$, and o directly calls some sentence that receives $\frac{1}{2}$ in $\mathcal{M}^\dagger_\beta$, o directly calls some sentence in H'. Therefore, because o was arbitrary, every element of H' calls some element of H'. Therefore $< R, H' >$ has no terminal nodes. So H' is an ungrounded set in $\mathcal{M}^\dagger_\beta$. Therefore $\mathrm{T}(c)$ is a member of an ungrounded set in $\mathcal{M}^\dagger_\beta$. Therefore, $\mathrm{T}(c) \in \mathrm{G}_-$ in $\mathcal{M}^\dagger_{\beta+1}$. By Theorem 3, the monotonicity conditions, $\mathrm{T}(c) \in \mathrm{G}^\dagger_-$. So the condition holds if p is $\mathrm{T}(c)$.

- Suppose p is $\neg\phi$. Suppose the inductive hypothesis holds for ϕ: if $v(\mathcal{M}^*)(\phi) = \frac{1}{2}$, then ϕ is a member of $\mathrm{G}^\dagger_-$. Suppose that $v(\mathcal{M}^*)(\neg\phi) = \frac{1}{2}$. By the semantics in Definition 4, $v(\mathcal{M}^*)(\phi) = \frac{1}{2}$. By inductive hypothesis, $\phi \in \mathrm{G}^\dagger_-$. Let α be the least ordinal such that $\phi \in G^{\dagger\alpha}_-$. α cannot be 0 or a limit ordinal, so there is an ordinal $\alpha - 1$. By the definition of the † construction, ϕ is a member of an ungrounded set in $\mathcal{M}^\dagger_{\alpha-1}$. Call that set B. The set $B \cup \{\neg\phi\}$ is also an ungrounded set. Therefore,

$\neg\phi \in G_{-}^{\dagger\alpha}$, and, by Theorem 3 (Monotonicity), $\neg\phi \in G_{-}^{\dagger}$. So the condition holds if p is $\neg\phi$.

- The proofs for the cases where p is $\phi \vee \psi$ or p $\exists v\phi$ follow the same pattern as the case where p is $\neg\phi$.

Right to Left: For all $\mathcal{L}$[T] sentences p, if p is a member of $G_{-}^{\dagger}$, then $v(\mathcal{M}^{*})(p) = \frac{1}{2}$.

- Suppose p is a member of $G_{-}^{\dagger}$. Now suppose for contradiction that p receives a classical value in $\mathcal{M}^{*}$. By Corollary 12.1, p has a classical value on $\mathcal{M}^{\dagger}$. But by Theorem 15, $v(\mathcal{M}^{\dagger})(p) = \frac{1}{2}$. Contradiction. So $v(\mathcal{M}^{*})(p) = \frac{1}{2}$.
- QED

There is, then, a straightforward parallelism between the * construction and the † construction. If $\mathcal{M}^{*}$ and $\mathcal{M}^{\dagger}$ are fixed-points constructed from base models that correspond to each other, then the $\mathcal{L}$[T] sentences in the anti-extension of G in $\mathcal{M}^{\dagger}$ will be *exactly* the sentences that are ungrounded in $\mathcal{M}^{*}$. Since the parallelism holds for all fixed-points, the standard axiomatization of T, PFK, is also sound for all $\mathcal{L}$[T,G] fixed-point models. (I won't pursue in this paper whether or not there is a natural axiomatization of G.[8])

4 Discussion

For an intuitive sense of how the G predicate works, it is helpful to think about the minimal $\mathcal{L}$[T,G] fixed-point. This is the fixed-point for the † construction starting with the base-model $\mathcal{M}[\text{T,G}]_0 = \mathcal{M}[\langle\varnothing, Q\rangle, \langle\varnothing, Q\rangle]$. (Where, again, Q is the set of objects in the domain that are not sentences of $\mathcal{L}$[T,G].) The fixed-point generated by this construction stands to Kripke's minimal fixed-point as $\mathcal{M}^{\dagger}$ stands to $\mathcal{M}^{*}$ in the previous section.

Let a be the $\mathcal{L}$[T,G] sentence $\neg\text{T}(< a >)$ and let b be $\text{T}(< b >)$. a is a Liar sentence and b is a truth-teller sentence. In the † construction beginning with $\mathcal{M}[\langle\varnothing, Q\rangle, \langle\varnothing, Q\rangle]$, a and b both get added to G_{-} in the first jump since they are members of ungrounded sets in the base model.[9] But there are other sentences that only get added to G_{-} at later stages—after their grounding possibilities "run out." Let p be some $\mathcal{L}$ sentence that receives 0 in the background interpretation, and let c be the sentence $\text{T} < p > \vee\neg\text{T}(< c >)$. Since c's first disjunct is settled-as-false in the base model, c can only receive $\frac{1}{2}$ in the fixed-point. But c is *not* a member of an ungrounded set in the base model. In any base model, the sentences in the following set all receive $\frac{1}{2}$: $\{c, \neg\text{T}(< c >), \text{T}(< c >), \text{T}(< p >)\}$. This

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[8] As a starting point, one might consider modifying Field's (2022) axiomatization of his "Strictly Classical" predicate *Str*. Unlike *Str*, however, G does not satisfy excluded middle.

[9] This is also true of any cycles or Yablo-paradoxical chains consisting solely of atomic alethic sentences.

set is closed under direct call among unevaluated sentences, but it is not an ungrounded set because $\mathrm{T}(< p >)$ is settled in the base model. However, $\{c, \neg\mathrm{T}(< c >), \mathrm{T}< c >\}$ will be an ungrounded set in the second stage, once $\mathrm{T}(< p >)$ receives the value 0.

In the minimal $\mathcal{L}$[T,G] fixed-point:

- if c is $\neg\mathrm{T}(< c >)$, c receives $\frac{1}{2}$ and $\neg G(< c >)$ and $\mathrm{T}(< \neg\mathrm{G}(< c >) >)$ both receive 1
- if d is $\neg\mathrm{T}(< d >) \vee \neg G(< d >)$, d receives $\frac{1}{2}$
- if e is $\mathrm{G}(< e >)$, e receives $\frac{1}{2}$
- if f is $\neg\mathrm{G}(< f >)$, f receives $\frac{1}{2}$

My suggestion is that G expresses groundedness in the same way that T expresses truth in standard Kripke constructions. T's extension contains everything true, though its anti-extension doesn't include everything untrue. G's extension contains everything grounded, though its anti-extension doesn't include everything ungrounded. Nonetheless, Theorem 16 shows that the anti-extension of G can be used to pick out all the things that are ungrounded in the fragment $\mathcal{L}$[T]. In that sense, I think it can serve to express in the object language the diagnostic insight of Kripke's original construction: that predicating truth induces semantic dependence, and that, to be grounded, you must be part of a non-defective semantic dependence structure.

4.1

One might object: the fact that the anti-extension of G does not include all sentences that are ungrounded disqualifies it from capturing ungroundedness in a fully general way. For instance, a Strengthened Liar, like d above, is ungrounded in the sense that it receives $\frac{1}{2}$ in any $\mathcal{L}$[T,G] fixed-point model, but that isn't captured by the semantics of G. d is not in the extension or anti-extension of G in any fixed-point. Moreover, the scope of G_{-} is narrowed in a way that might seem unprincipled. In my introduction, I suggested that G should arguably be susceptible to semantic failure precisely because it is a semantic predicate like T. But the characterization of ungrounded sets in Definition 9 is based on a analysis of semantic dependence (direct calling) according to which predications of T initiate semantic dependence links but predications of G do not. In other words, G_{-} is not tracking ungroundedness *per se*, just ungroundedness induced by predications of truth. This threatens to make G uninteresting. We already knew how to express "ungrounded in $\mathcal{L}$[T]" in a richer language that extends $\mathcal{L}$[T]—how does the G predicate improve on this?

In addressing these objections, we should be clear about the problems G is (and is not) intended to solve. It is not intended to be a general response to "Revenge Paradoxes" or to remove all philosophically puzzling expressive limitations present in Kripke's theory of truth. As I said in the introduction, I'm forswearing the project of introducing a "ungroundedness" predicate to that correctly applies to all sentences of that language that are ungrounded. Whatever pernicious Revenge objections can be raised against Kripke's theory of truth can equally be raised against my semantics for $\mathcal{L}$[T,G].[10] But I do think the inclusion of the G predicate allows us to express, in the object language, what I have characterized as Kripke's main diagnostic insight.

The objection above is quite right that I have described a groundedness predicate whose anti-extension is specifically sensitive to networks of semantic dependence induced by predications of *truth.* This is a restriction—it doesn't take account of the semantic dependence induced by predications of G—but it is not an unmotivated restriction. Arguably, truth and falsity are *basic* semantic categories and the dependence among claims attributing truth or falsity is the *basic* sort of semantic dependence. It is the kind of semantic dependence that Kripke's diagnosis is concerned with. And although G_{-} does specifically track ungroundedness due to predications of truth, it does *not* specifically track ungroundedness in the language $\mathcal{L}$[T]. Infinitely many sentences containing occurrences of G will end up in G_{-} in the fixed-point models I've described—including sentences commenting on the *truth* of sentences containing G. For instance, let s be an $\mathcal{L}$ sentence that receives 0 in the background interpretation, and let h be $\neg\mathrm{T}(< h >) \vee \neg\mathrm{G}(< s >)$. $\neg\mathrm{G}(< h >)$ will get the value 1 in any $\mathcal{L}$[T,G] fixed-point model.[11] So G is not merely a tool for expressing semantic failures occurring in the language $\mathcal{L}$[T] in a richer metalanguage—it is a tool for expressing *in* $\mathcal{L}$[T,G] Kripke's diagnostic insight as it applies to $\mathcal{L}$[T,G].

4.2

Now I will compare my G predicate to some of the object-language semantic vocabulary introduced in recent work by Lucas Rosenblatt and collaborators.[12] Lucas Rosenblatt

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[10] It's also worth noting that, although one can truly assert $\neg\mathrm{G}(< \lambda >)$, where λ is a Liar sentence, one cannot truly assert $\neg\mathrm{T}(< \lambda >)$. So the resulting system does not allow for the expression of the Chrysippus Intuition—the suggestion that we can truly say of pathological sentences that they are not true.

[11] Once s is added to G_{+}, $\neg\mathrm{T}(< h >) \vee \neg\mathrm{G}(< p >)$ will be a member of the ungrounded set $\{h, \neg\mathrm{T}(< h >), \mathrm{T}(< h >)\}$ and it will get added to G_{-} in the next step of the sequence.

[12] Since Kripke's original publication, a number of authors have made proposals regarding object-language tools for capturing something similar to Kripke's notion of groundedness/ungroundedness, e.g. the hierarchy of pathology predicates in 2007, Tourville and Cook 2016, 2020; the "strongly classical" predicate in Field 2022. I limit myself to commenting on work by Rosenblatt, Gallovich, and Szmuc for reasons of space, and because the comparison with my G predicate can be made quite directly.

(2021, 2022), alone and in collaboration with Camila Gallovich (2022, 2022) and Damián Szmuc (2014), has made a number of proposals about how to add predicates (or operators) to the object language of a Kripkean theory of truth that allow one to express semantic notions like *groundedness* and *paradoxicality.* All of these works share a core that involves constructing fixed-point models for the enriched language where, at each step of the construction, the valuation of a sentence containing the new semantic vocabulary is determined by considering a variety of fixed-points *extending* previous models. For a direct comparison with my account, I'll provide a (largely informal) summary for the proposal in Rosenblatt 2022, which concerns an object language groundedness predicate. I'll call that predicate G^{Ros} and the language that extends $\mathcal{L}[\mathrm{T}]$ by including it $\mathcal{L}[\mathrm{T},\mathrm{G}^{Ros}]$.

The fixed-point construction for $\mathcal{L}[\mathrm{T},\mathrm{G}^{Ros}]$ begins with Kripke's minimal fixed-point for $\mathcal{L}[\mathrm{T}]$ as the base model, v_0, where the extension and anti-extension of G^{Ros} are both empty. For any valuation v_i, we use the notation $\mathcal{V}_i$ to refer to the set of all K3 fixed-point valuations that extend v_i. Rosenblatt provides an interpretation for G^{Ros} by defining a sequence of valuations, beginning with v_0, that arrives at a fixed-point. The clause of Rosenblatt's definition for a successor ordinal $\alpha + 1$ is :

$$v_{\alpha+1}(\mathrm{G}^{Ros}(<\phi>)) = \begin{cases} 1 & \text{if } \forall v \in \mathcal{V}_\alpha : v(\phi) = 1 \text{ or } \forall v \in \mathcal{V}_\alpha : v(\phi) = 0 \\ 0 & \text{if } \forall v \in \mathcal{V}_\alpha : v(\phi) = \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

At $v_{\alpha+1}$, all sentences that receive a classical value in every fixed-point extending v_α get (in effect) added to the extension of G^{Ros}, all sentences that receive $\frac{1}{2}$ in every fixed-point extending v_α get (in effect) added to the anti-extension of G^{Ros}, and G^{Ros} remains undefined on everything else. The clause for a limit ordinal τ is:

$$v_\tau(\mathrm{G}^{Ros}(<\phi>)) = \begin{cases} 1 & \text{if } \forall v \in \bigcap\{\mathcal{V}_\beta : \beta < \tau\} : v(\phi) = 1 \text{ or } \forall v \in \bigcap\{\mathcal{V}_\beta : \beta < \tau\} : v(\phi) = 0 \\ 0 & \text{if } \forall v \in \bigcap\{\mathcal{V}_\beta : \beta < \tau\} : v(\phi) = \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

In a way, Rosenblatt's construction takes an opposite perspective to the one in this paper—Rosenblatt and collaborators characterize sentences by "looking up" and considering their behavior at different potential fixed-points, whereas, in my construction, we "look down" to see whether the dependence chains linking sentences bottom out appropriately. But extensionally they cover much of the same ground.[13] In Rosenblatt's fixed-points, the extension of G^{Ros} contains exactly the sentences that receive classical

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[13] Interestingly, both take inspiration from work by Stephen Yablo: 2003 (in Rosenblatt's case) and 1982.

values, just as the extension of G does. But there is a significant difference between the anti-extension of the two predicates. The anti-extension of G^{Ros} at one of Rosenblatt's fixed-points v_γ contains all and only the sentences that receive $\frac{1}{2}$ at *every* fixed-point extending v_γ. This corresponds to Kripke's concept of a *paradoxical* sentence, which is narrower than the concept of an ungrounded sentence. For instance, b, the truth-teller sentence we introduced above is ungrounded but not paradoxical. In all of Rosenblatt's fixed-points $\mathrm{G}^{Ros}(< b >)$ receives the value $\frac{1}{2}$. b receives $\frac{1}{2}$ in v_0, Kripke's minimal fixed-point; and any successor stage $\alpha + 1$ in the construction, there are fixed-points extending v_α in some of which b receives 1 and in some of which b receives 0. So it remains evaluated as $\frac{1}{2}$ in every stage.[14] By contrast, as we said, b will be in the anti-extension of G in the minimal $\mathcal{L}$[T,G] fixed-point, since it is a member of an ungrounded set in the base model. So although the anti-extension of Rosenblatt's G^{Ros} provides a useful tool for expressing paradoxicality in the object language, it is not successful as a tool for expressing ungroundedness. In fact, in work where Rosenblatt (2021) and Gallovich (2022, 2024) introduce a *paradoxicality* predicate, it is defined simply by flipping the cases for 0 and 1 in the clauses above.

This difference is not a superficial one—there are intrinsic difficulties with collecting the ungrounded-but-non-paradoxical sentences using Rosenblatt's "looking up" approach. To put it intuitively, at a successor stage in the construction, when one looks up to the fixed-points that extend the previous valuation, there is no obvious basis for distinguishing ungrounded-but-non-paradoxical sentences from sentences featuring the new semantic vocabulary whose interpretation is in progress—the valuation of *both* can vary across fixed-points. Rosenblatt and Gallovich acknowledge this problem in their 2024, where they attempt to describe an object language that contains a paradoxicality operator as well as a "hypodoxicality" operator. (A "hypodox," in Rosenblatt and Gallovich's terminology, is a sentence, like b, that is true in some Kripkean fixed-points and false in others.[15]) Their solution is to restrict the truth-conditions of sentences containing the hypodoxicality operator $\mathcal{O}^H$ such that $\mathcal{O}^H\phi$ is only true if ϕ is a sentence of $\mathcal{L}$[T], rather than a sentence of the enriched language containing $\mathcal{O}^H$. This is a significant short-coming: the operator really expresses hypodoxicality-in-$\mathcal{L}$[T].[16] No such restriction

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[14] Taking a more general perspective, it would be more precise to say that $\mathrm{G}^{Ros}(< b >)$ never receives the value 0 in any of Rosenblatt's fixed-points. Rosenblatt typically stipulates that his fixed-points are constructed from a sequence that begins with Kripke's *minimal* fixed-point, but, in principle, one could start with a non-minimal fixed-point in which b receives a classical value. In that case, $\mathrm{G}^{Ros}(< b >)$ will receive the value 1 in the resulting $\mathcal{L}[\mathrm{T},\mathrm{G}^{Ros}]$ fixed-point. Either way, b never ends up in the anti-extension of G^{Ros}.

[15] They attribute this terminology to Peter Eldridge-Smith 2007.

[16] A similar issue arises for the hierarchy of *pathological* predicates/operators described in Cook 2007 and Tourville and Cook 2016, 2020. For reasons of space, I cannot give an adequate technical summary of such works, but to characterize the problem roughly: in those works a sentence of the form $P_\alpha\phi$, where P_α is the α-th pathology operator, can only be true if ϕ is a sentence of some language $\mathcal{L}_{\beta<\alpha}$, where P_α does not occur.

is necessary my framework: in the minimal $\mathcal{L}$[T,G] fixed-point the anti-extension of G contains many hypodoxes and other ungrounded-but-non-paradoxical sentences that are sentences of $\mathcal{L}$[T,G] proper. For instance, if f is the sentence $\mathrm{T}(< f >) \vee \neg G(< p >)$, where p is a sentence of $\mathcal{L}$, then f will be in the anti-extension of the G in $\mathcal{M}^{\dagger}$.[17]

I'll close by making a suggestion for future research. At a couple points in this paper, I have pointed out that my semantics for $\mathcal{L}$[T,G] has no pretension to be a general response to revenge paradoxes. But it is natural to wonder how it might fit in to such a response. One path worth exploring would be a modification of the view Tourville and Cook call "Embracing Revenge."[18] On this view, the perpetual availability of revenge paradoxes is due the fact that the concept of a truth-value is indefinitely extensible. On such a picture, revenge paradoxes are not a "problem to be overcome," they are a natural consequence of the fact that no language can have a fully exhaustive classification of truth-values. I think the reasons Cook and Tourville offer for thinking that the notion of a *truth-value* is indefinitley extensible equally support the idea that we have an indefinitely extensible notion of *semantic dependence.* I won't pursue the suggestion here, but there is no obvious obstacle to the project of defining a sequence of languages $\mathcal{L}[\mathrm{T},\mathrm{G}_\alpha]$, such that every successor language contains a groundedness predicate, modeled on G, that is suitable for describing networks of dependence induced by the semantic vocabulary in the prior language. If one is compelled by the "Embracing Revenge" response, one might fruitfully develop it in the setting of a hierarchy of languages based on $\mathcal{L}$[T,G].

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[17] I should note that my G predicate is not suited for the purposes Rosenblatt and Gallovich have in their 2024 paper, which is specifically to *distinguish*, in the object language, between paradoxical and hypodoxical sentences. But since Yablo 1982 has shown that Kripke's notion of paradoxicality can be given an alternative analysis in dependence-graph-theoretic terms, it seems quite likely that one could add a predicate to my construction that specifically tracks ungrounded sets consisting of paradoxes or hypodoxes. I leave this project to future work.

[18] This view is defended in Cook 2007; Tourville and Cook 2016, 2020; Schlenker 2010.

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