

MIDDLE SCHOOL EXPLORERS

ENTRANCE TEST



Course Objectives and Test Guidelines

Middle School Explorers (MSE) is an advanced course for students in grade 5-7. The goal of this course is to provide a strong foundation for young students, preparing them for advanced math competitions such as the Canadian Math Olympiad (CMO), USA Math Olympiad (USAMO), and International Math Olympiad (IMO). By participating in this course, students will not only enhance their problem-solving skills but also gain a deeper understanding of mathematical concepts, which will be beneficial for their academic and personal growth.

This course is designed to minimize repetition and maximize exploration of a wide variety of problem-solving concepts. Students will be encouraged to spend time on each problem, understanding that the learning process is as important as the solution. The course will also focus on developing your skills in writing proofs, a crucial aspect of advanced math competitions.

The test for this course features a few complex problems. It's important to note that we're not expecting students to solve all the problems. What we're really interested in is how students approach the problems and demonstrate their problem-solving skills. Therefore, detailed solutions are expected, regardless of whether the problem was fully solved.

It's important to understand that this course will not be easy. However, by setting this expectation, we aim to prepare you to navigate through complex concepts with patience and determination. We are looking for students who are self-motivated and have strong fundamentals. While we won't be covering basic concepts in this course, such as prime factoring, we believe that with the right mindset and dedication, you can excel in this advanced course.

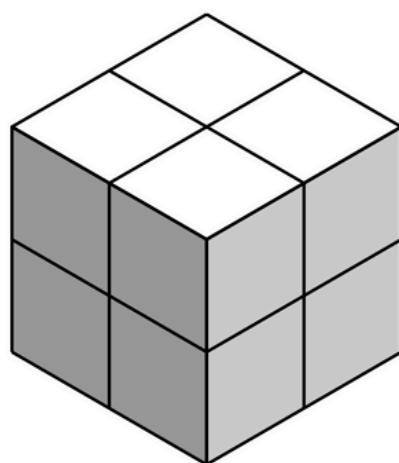
This course will be taught by a highly experienced and dedicated coach, who not only focuses on math but also help students in developing right mindset and habits. His goal is to develop the best talent in Canada, and he is committed to providing the best learning experience for all students.

If you are still interested in the course then write detailed solutions for questions you can solve and **email your work as a single pdf document to info@ckstem.org before 31st July.**

1. MAKING CUBES



Tina enjoys constructing various objects with her box of centicubes (1 cm cubes) by placing the cubes face to face. For example, she could use eight centicubes to make a solid cube of side length 2 cm.



- a How many centicubes will Tina need to construct a collection of three solid cubes of side lengths 3 cm, 4 cm, and 5 cm?
- b Tina used all the centicubes in her box to construct a collection of five cubes of side lengths 1 cm, 2 cm, 3 cm, 4 cm, and 5 cm. She takes the five solid cubes apart to have individual centicubes. Show how she can rearrange all the individual centicubes to construct a collection of fewer than five solid cubes.
- c Tina increases the number of centicubes in her box to 441. Show how she can arrange all her centicubes into a collection of six solid cubes and how she can rearrange all her centicubes into a collection of four solid cubes.
- d Tina adds to her box of 441 centicubes all the centicubes from a solid cube of side length 7 cm. Show how she can rearrange all these centicubes into a collection of seven solid cubes, how she can rearrange all these centicubes into a collection of five solid cubes, and how she can rearrange all these centicubes into a collection of four solid cubes.

2. ZIRN CRUNCH



The ingredients in Nino's recipe for Zirn Crunch are 20 grams of zirn, 90 grams of honey, and 140 grams of cereal.

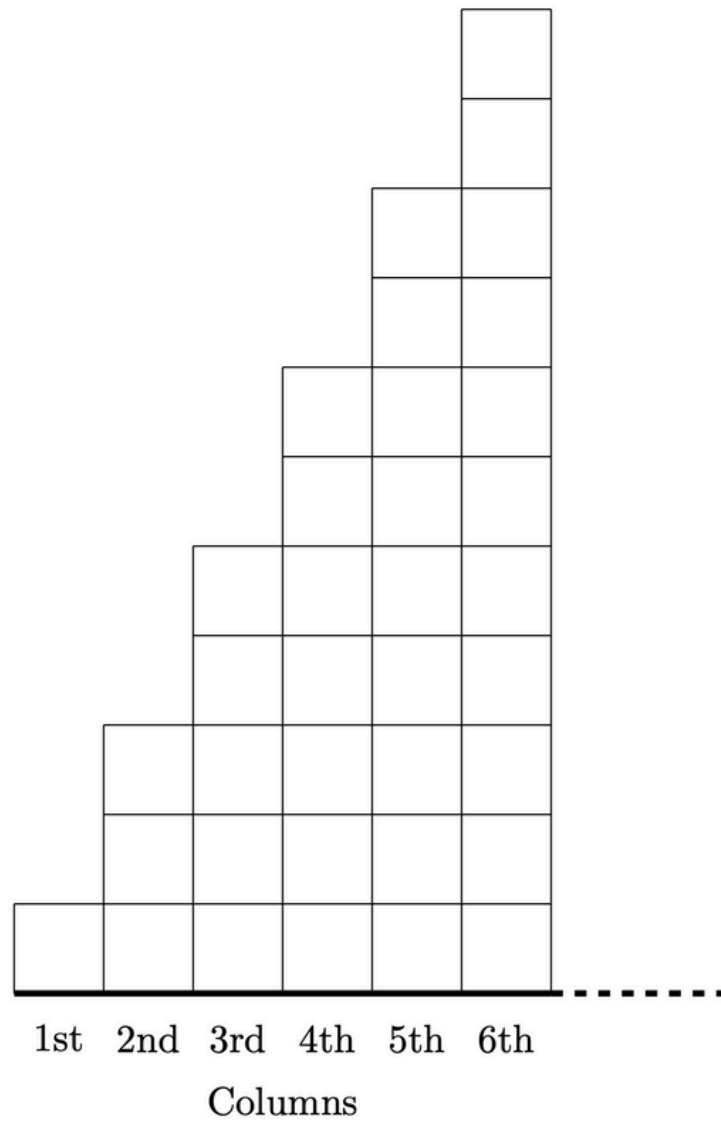
After he mixed the zirn and honey together, Nino realised he made a mistake when he used 20 grams of honey with 90 grams of zirn. Instead of wasting this mixture, he decided to make a bigger batch of Zirn Crunch without adding any more zirn.

- a Calculate how much extra honey Nino needed to add to make the bigger batch of Zirn Crunch.
- b How much cereal was required to make the bigger batch and how much Zirn Crunch would this have made?
- c Nino bought a 1-kilogram box of cereal and decided to use it all in a fresh batch of Zirn Crunch. How much Zirn Crunch would this make (to the nearest gram)?
- d How much of each ingredient would there be in a 50-kilogram batch of Zirn Crunch?

3. STEPS TO INFINITY

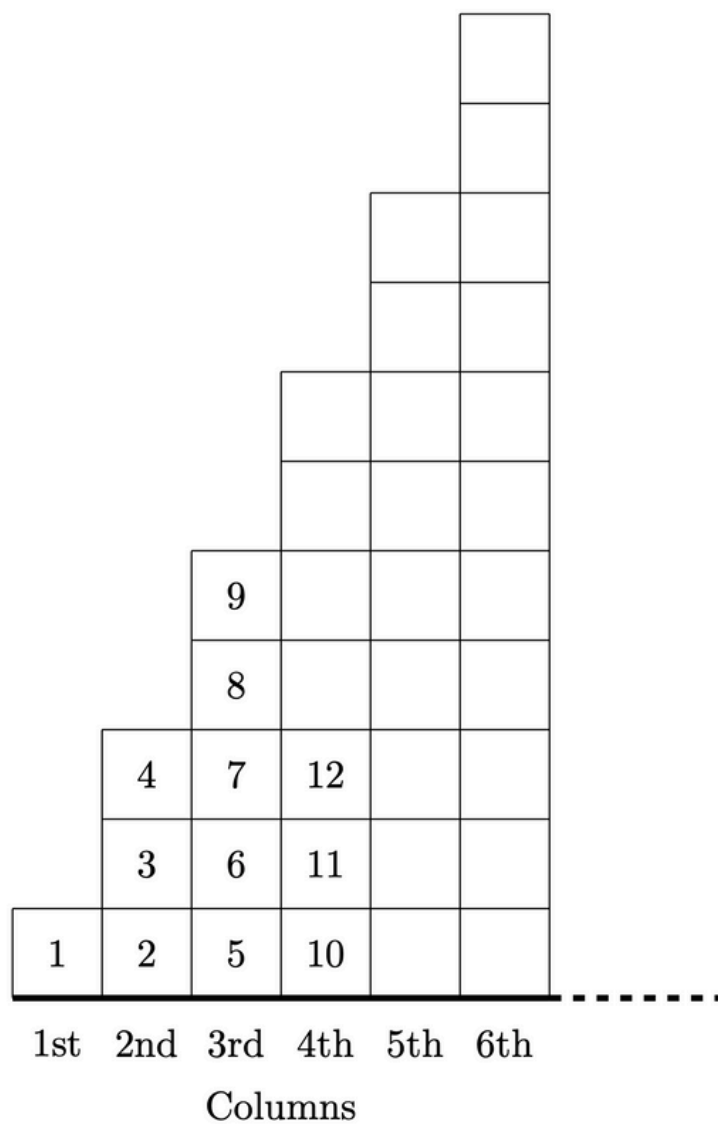


Each student in the class is given a diagram of a staircase like this:



Some students add extra rows at the bottom of this blank staircase so that each added row has its leftmost square in column 1. They then write the numbers 1, 2, 3, etc. in the squares, starting with 1 in the bottom-left square of their staircase and moving up each successive column without missing any squares.

a Ahmed doesn't add any extra rows and places the numbers in the squares of his blank staircase as shown.

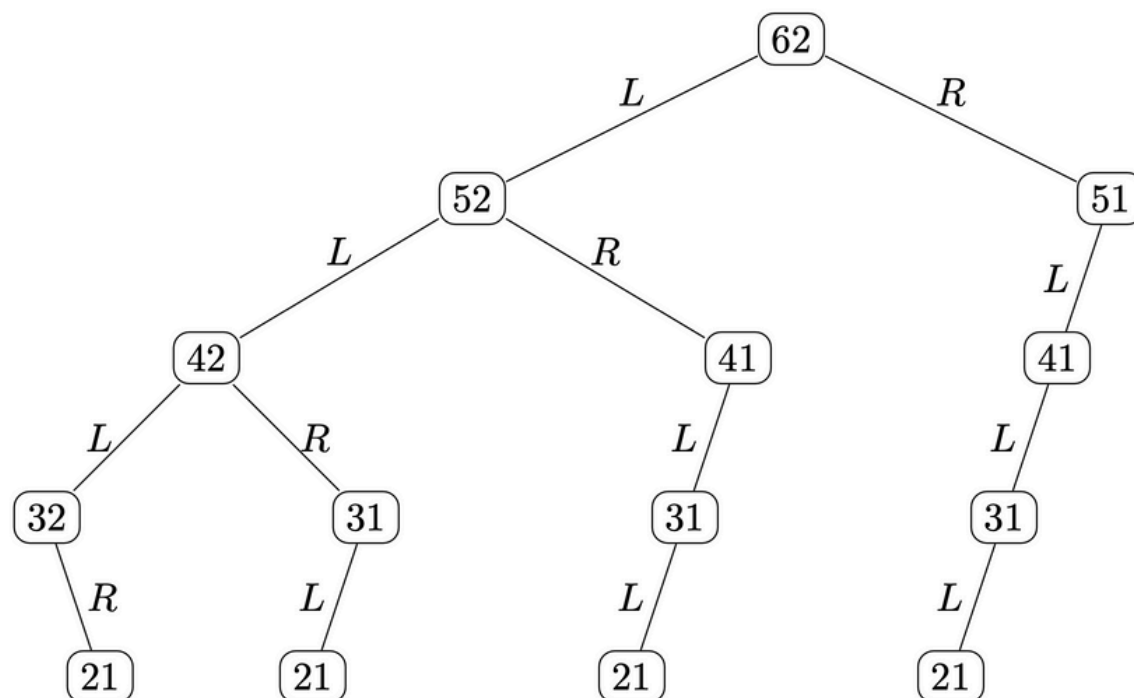


Continuing Ahmed's pattern, one of the columns would have the number 145 in its bottom square. What would be the top number in that column?

4. DROP BEARS



Drop bears live in nests in drop bear trees. Each nest has a number. The nest with the largest number is called *home* and its number is used to name the tree. For example, the following diagram shows drop bear tree 62.



There are at most two downward branches from each nest to nests immediately below it. Each branch is labelled L (for left) or R (for right), depending on its direction when looking at the tree. Every drop bear knows how to get to its nest from home by selecting the correct sequence of Ls and Rs, called an LR-path. For example, the first 21 nest from the left in tree 62 has LR-path LLLR and the second 21 nest has LR-path LLRL.

The nest numbers are called *addresses*. They consist of two positive integers a and b written next to each other, with a on the left. We insist that b is a single digit, a has at most two digits, and $a > b$. If a has two digits, then its first digit is not 0. For example, 62, 503, and 758 are valid addresses but 40, 44, and 062 are not.

Drop bears do special subtraction to work out the number of their nest. From any nest whose address has first integer a and second integer b , to get the address of the nest below and to the left they subtract 1 from a , and to get the address of the nest below and to the right they subtract 1 from both a and b . For example, the left and right nests below 214 are 204 and 203 respectively.

- Draw drop bear tree 148 but only down to all the nests with addresses larger than 110.
- In drop bear tree 148, what is the address of the nest at the end of the LR-path RLRL?
- List the LR-path in drop bear tree 148 to each nest with address 107.
- How many nests with address 95 are there in drop bear tree 148?

5. TRIBONACCI SEQUENCES



A *tribonacci sequence* is a sequence of numbers such that each term from the fourth onwards is the sum of the previous three terms. The first three terms in a tribonacci sequence are called its *seeds*. For example, if the three seeds are 6, 19, 22, then the next few terms are 47 ($6 + 19 + 22$), 88 ($19 + 22 + 47$), 157 ($22 + 47 + 88$), and 292 ($47 + 88 + 157$).

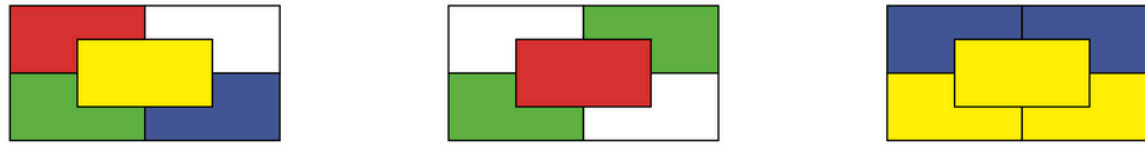
- a Find the smallest 5-digit term in the sequence above.
- b The 5th, 6th, 7th terms of a tribonacci sequence are respectively 36, 71, 135. What are the three seeds for this sequence?
- c The seeds of a tribonacci sequence are 20, 17, 2017. Is the 2017th term even or odd? Explain.
- d If a tribonacci sequence has 20 as its second seed and 17 as its third seed, find all positive integers that can be its first seed so that 2017 appears as a term somewhere in the sequence.

6. MANY FLAGS

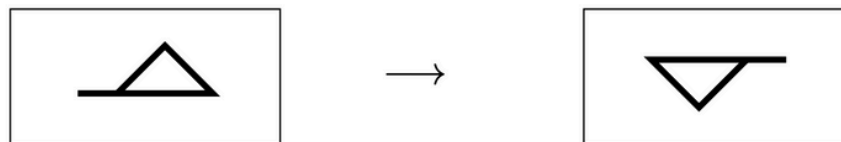


Consider a flag which is divided into five regions like those shown below. Up to five colours, red, white, green, blue, yellow, are used to colour the regions. Each region has only one colour. Two, three, four, or five regions may have the same colour.

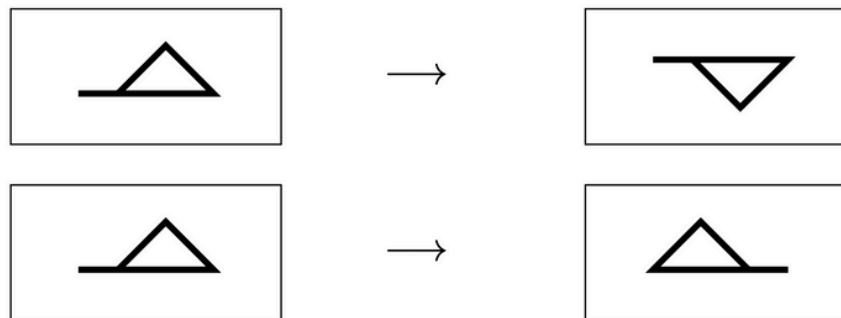
Each flag is designed so that it can be hung horizontally on a wall. The regions are shaped so that, when all regions have the same colour, the flags are symmetrical both horizontally and vertically. Here are three examples:



- a How many flags have four colours with the top-left and bottom-right colours both red?
- b How many flags have five different colours?
- c How many flags look the same if rotated 180° on the wall like this?



- d How many flags look the same if flipped top to bottom or side to side like this?



7. FACTOR CHAINS



A *factor chain* is an ordered list of *different* whole numbers with the following property: for each pair of adjacent numbers, one of the numbers is a factor of the other. The *length* of a factor chain is the number of numbers it contains.

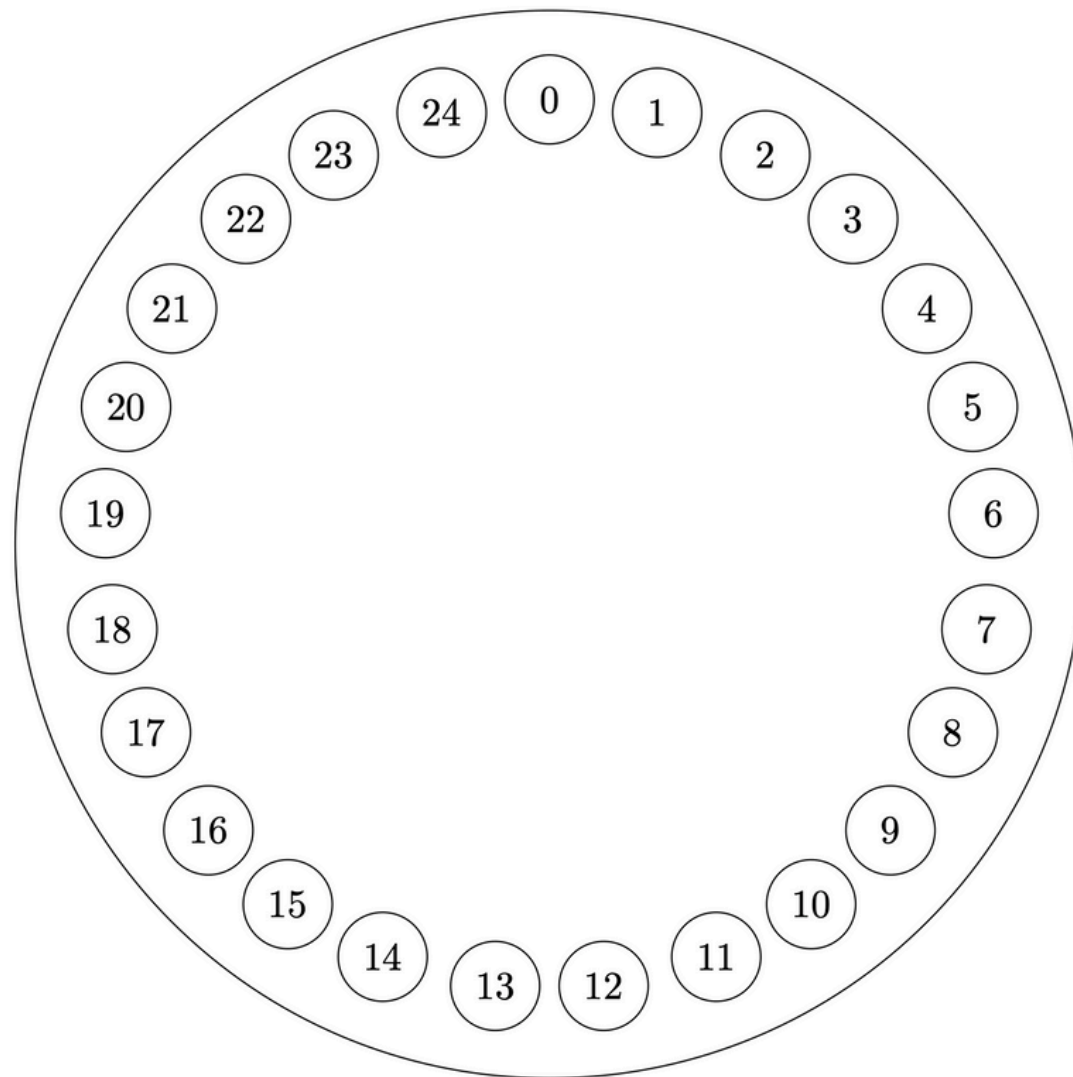
For example, using only numbers from 1 to 8 we can make the factor chain $8 \rightarrow 4 \rightarrow 1 \rightarrow 6 \rightarrow 3$ of length 5, or the chain $4 \rightarrow 2 \rightarrow 6$ of length 3, etc. Pairs of adjacent numbers, such as 8 and 4, are called *neighbours*. Other *possible neighbours* are 1 with 7, 2 with 8, etc.

- a Use all the numbers from 1 to 6 to make a factor chain of length 6.
- b For a factor chain using only numbers from 1 to 20, which two numbers have exactly six possible neighbours each?
- c Find a factor chain of length 17 using only numbers from 1 to 20.
- d Explain why no factor chain using only numbers from 1 to 20 can have length more than 17.

8. CIRCLE HOPSCOTCH



A hopping circuit is painted on a school playground pavement. It consists of 25 small circles arranged in a large circle and numbered 0 to 24.



Each student starts at 0 and hops clockwise either 3 places (a 3-hop) or 4 places (a 4-hop) on each turn. For example, a student's first hop from 0 will end on either position 3 or 4. Students must go twice around the circuit and end back at 0 to complete a game. All students list in order the numbers they land on and record the total number of hops they take.

- In one game a student took 13 hops. Write down a possible list of numbers he landed on.
- Find all possible combinations of the number of 3-hops and the number of 4-hops in a game.
- What is the smallest number of different numbers a student can land on in one game? Explain your answer.
- Jo and Mike decide to play a longer version of the game according to the following rules. They take each of their hops at the same time starting with both on 0. Whenever Jo takes a 4-hop, Mike takes a 3-hop; whenever Jo takes a 3-hop, Mike takes a 4-hop.

Jo's first five full hops on each lap are 4-hops. After that, she takes 3-hops until she next reaches or passes 0. How many laps will each of them have completed when they next meet at 0?

9. POINTY NUMBERS



A *pointy number* can be either upward pointy or downward pointy. An upward pointy number has digits in ascending order up to a digit called the *point* from which the remaining digits are in descending order. A downward pointy number has digits in descending order to a point digit from which the remaining digits are in ascending order. In all pointy numbers:

- If digits are adjacent, then they differ by 1.
- The first and last digits are the same and not 0.

For example:

- 34543 is an upward pointy number
 - 434 is a downward pointy number
 - 24642 is not a pointy number as adjacent digits differ by 2
 - 654345 is not a pointy number as the first and last digits are different
 - 789987 is not a pointy number since it does not have a single point
 - 012343210 is not a pointy number as it starts and ends with 0.
- a** Explain why all 7-digit pointy numbers that are divisible by 3 must have their first digit divisible by 3.
- b** How many pointy numbers are divisible by 6?
- c** An upward pointy number is added to a downward pointy number with the same number of digits. Show that the sum cannot be a prime number.
- d** How many pairs of 7-digit upward and downward pointy numbers when added give a sum that is a palindrome?

10. UNEQUAL PARTITIONS



A partition of an integer into distinct positive parts (no two are equal) is called an *unequal partition*. For example, there are seven unequal partitions of 12 into 3 parts: $12 = 9 + 2 + 1 = 8 + 3 + 1 = 7 + 4 + 1 = 7 + 3 + 2 = 6 + 5 + 1 = 6 + 4 + 2 = 5 + 4 + 3$.

The *span* of a partition is the difference between the smallest and largest number in the partition. For example, the spans of the unequal partitions of 12 into 3 parts listed above are respectively 8, 7, 6, 5, 5, 4, 2. Thus 2 is the smallest span of an unequal partition of 12 into 3 parts.

- a Find at least two unequal partitions of 2017 into 5 parts with a span of 7.
- b Find an unequal partition of 2017 into 5 parts with the smallest possible span.
- c Find the smallest number that has an unequal partition into 5 parts. Describe the set of all numbers that have an unequal partition into 5 parts with a span of 4.
- d Show that for each number that has an unequal partition into 5 parts, the smallest span for such partitions is 4 or 5.

11. ROWING MACHINE



A person exercising on a rowing machine is kept informed about his/her progress by various numbers displayed by the machine. Three numbers of particular importance are the total distance rowed in metres, the total time rowed in minutes and seconds, and the current pace, which is the time it would take in minutes and seconds to row 500 metres.

For example, the diagram shows that this rower has covered a distance of 1000 metres, has been rowing for 5 minutes and 50 seconds, and would take 2 minutes and 40 seconds to row 500 metres at the current pace.

Distance	1000
Time	5:50
Pace	2:40

- a Thomas and Jack start rowing at the same time. Each aims to get to 1000 m first. Thomas' strategy is to row the whole distance at a 3 minute pace. Jack decides to row at a slower 3 minute 20 second pace for the first 3 minutes and then row faster at a 2 minute 40 second pace for the rest of the distance. Who gets to 1000 m first?
- b Thomas and Jack repeat the task of rowing to 1000 m first. Thomas maintains his plan of rowing 1000 m at a 3 minute pace. Jack changes his strategy to row the first 600 m at a 3 minute 20 second pace, then the next 400 m at a 2 minute 40 second pace. Who gets to 1000 m first?
- c Jack plans to row 3000 m in 15 minutes. He sets out at a 2 minute 38 second pace, but after rowing 600 m realises that he will not be able to cover 3000 m in 15 minutes at his current pace. At what pace does he need to row the remainder of his training session to reach his goal?
- d Thomas rows 1000 m at a 3 minute pace. Jack rows at a 3 minute 20 second pace at the beginning, and then he rows at a 2 minute 40 second pace. For what distance should Jack row at a 3 minute 20 second pace to finish the 1000 m at the same time as Thomas?

12. MEANS AND MEDIANS

To compare the mean and median of a set of positive integers, we calculate its *mean-median ratio* (or MMR), that is, the mean divided by the median. Here are two examples.

The set $\{1, 6, 17\}$ has mean 8 and median 6, so its MMR is $\frac{8}{6} = \frac{4}{3}$.

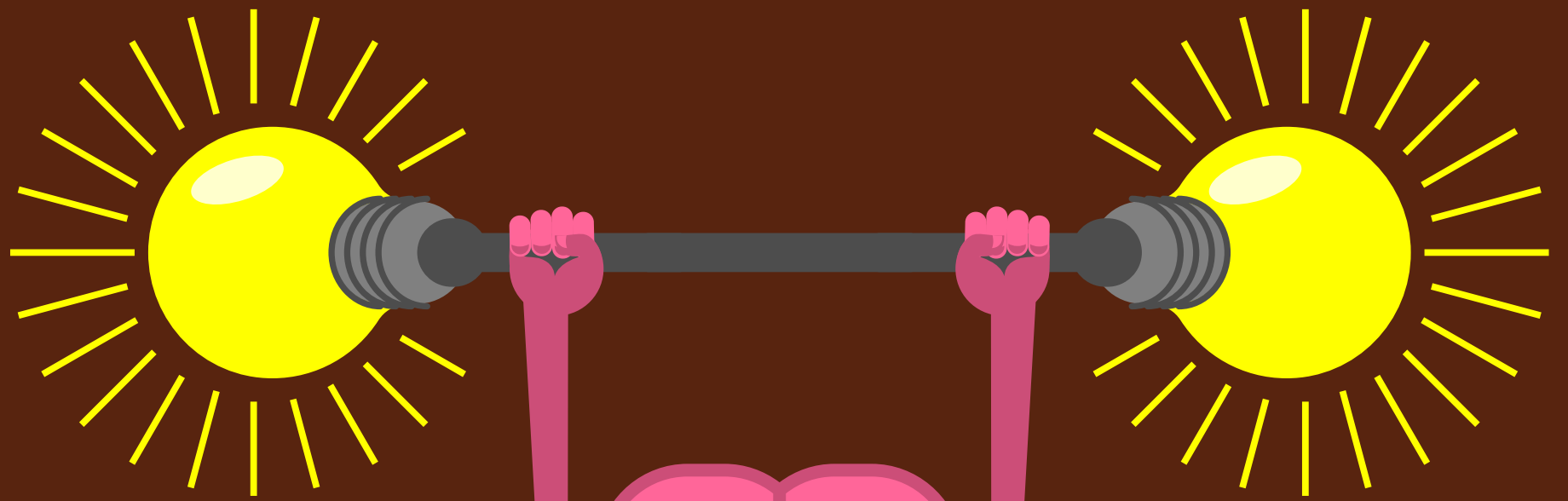
The set $\{8, 10, 15, 15\}$ has mean $\frac{48}{4} = 12$ and median $\frac{10+15}{2} = \frac{25}{2}$, so its MMR is $12 \div \frac{25}{2} = \frac{24}{25}$.

- a Find two sets of three positive single-digit numbers with MMR equal to $\frac{4}{5}$.
b Show that every set of three positive integers has MMR greater than $\frac{2}{3}$.

The MMR of a set of three positive integers can be as close to $\frac{2}{3}$ as we like. One way to see this is to consider the set $\{3, n, n\}$. It has mean $\frac{3+2n}{3}$ and median n , so its MMR is $\frac{2}{3} + \frac{1}{n}$. Making n larger and larger brings the MMR closer and closer to $\frac{2}{3}$.

Since the MMR of every set of three positive integers is greater than $\frac{2}{3}$ and the MMR of a set of three positive integers can be as close as we like to $\frac{2}{3}$, we say $\frac{2}{3}$ is the MMR *threshold* for sets with three positive integers.

- c Find the MMR threshold for sets of four positive integers. Carefully justify your answer.



Thank



YOU

