Response Analysis of First-Order Transfer Functions

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1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the Response Analysis of First-Order Transfer Functions Learning Guide*

ID	Learning Outcome	Achieved
1.	Convert a first-order transfer function into standard form and identify its system gain and time constant.	☐
2.	Explain how the system gain determines the final value of the output, and how the time constant determines the speed of the transient response.	☐
3.	Sketch and interpret the response of a first-order system to a step input, including the steady-state error (SSE) of a closed-loop control system.	☐
4.	Explain how system parameters and the proportional control gain affect the response, using practical examples such as an RC low-pass filter and water tank level control.	☐

2.0 Introduction

The Inverse Laplace Transform and Final Value Theorem Learning Guides showed how to determine the time solution of a system and its final value in steady state. Many engineering systems, such as an RC circuit, a water tank or a thermal process, are described by a first-order transfer function. For these systems, there is a much quicker way to understand the whole response, both its transient part and its final value, without solving for the time solution each time.

This Learning Guide introduces the standard form of a first-order transfer function. The standard form has two parameters: the system gain, K, which determines the final value of the output, and the time constant, τ, which determines how quickly the output responds. Once these two values are known, the response can be sketched immediately, and for a closed-loop control system the output can be compared with the reference to determine the steady-state error (SSE).

The guide begins with two practical examples, an RC low-pass filter and a water tank, where MATLAB and Simulink are used to investigate how the system parameters and the proportional control gain, K_p, change the response. The standard form is then introduced and used to explain these observations.

Why use the First-Order Standard Form?

The standard form of a first-order transfer function is useful because:

i. The system gain, K, gives the final value of the output once the transients have decayed.
ii. Once the final value is known, the SSE of a closed-loop control system can be determined as the difference between the reference and the output.
iii. The time constant, τ, describes the transient part of the response, that is, how long the output takes to reach steady state.
iv. Together, K and τ give a quick way to sketch the response and to check it against the requirements.

3.0 First-Order System Responses

The Final Value Theorem Learning Guide considered only the final value of the output. In this section, the transient part of the response is also considered. Practical examples are used to show how the system parameters and the proportional control gain, K_p, affect both the transient part of the response and the final value.

RC Circuit (Low-Pass Filter)

An RC low-pass filter is a simple electrical circuit, shown in Figure 1. When a voltage is applied at the input, the capacitor charges through the resistor;

when the voltage is removed, the capacitor discharges through the resistor. The circuit is described by the ordinary differential equation (ODE):

$$V_{in}(t) = RC\,\frac{dV_{out}(t)}{dt} + V_{out}(t) \tag{1}$$

where $V_{in}(t)$ is the supply (input) voltage, R is the resistance, C is the capacitance and $V_{out}(t)$ is the output voltage across the capacitor. Applying the Laplace transform to Equation (1) and rearranging, as covered in the Laplace Transforms Learning Guide, gives the transfer function:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1} \tag{2}$$

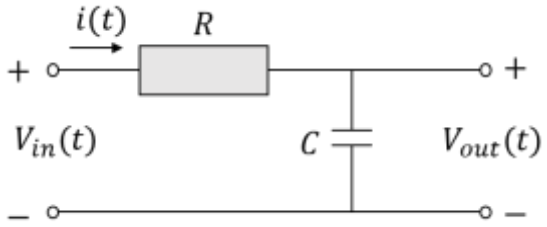


Figure 1: *RC Circuit (Low-Pass Filter)*

The RC circuit is called a low-pass filter because it passes low-frequency signals and attenuates high-frequency signals. High-frequency noise on a measurement is undesirable in a control system, because it can lead to poor control. Figure 2 shows a noisy measured signal and the same signal after it has passed through an RC low-pass filter, with $RC = 1$. The MATLAB and Simulink files for this figure are given in Appendix A.

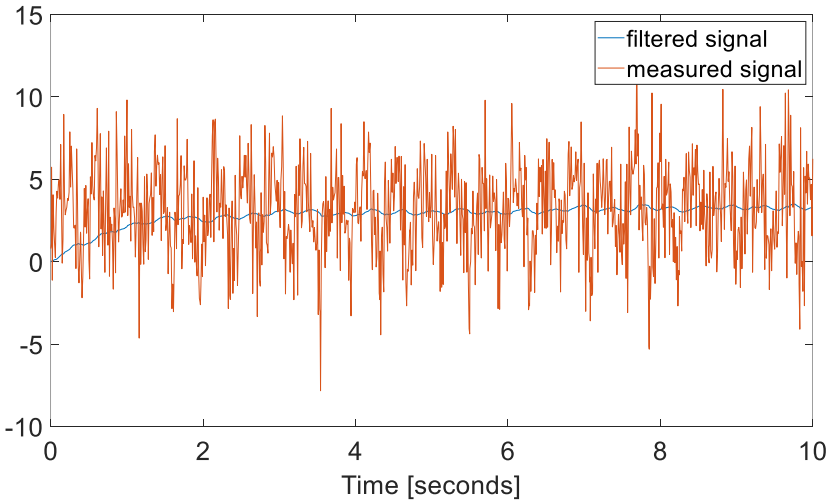


Figure 2: *A Noisy Measured Signal Smoothed by an RC Low-Pass Filter*

A drawback of the RC low-pass filter is that it introduces a lag, which is visible at the start of the filtered signal in Figure 2. During this time, the filtered signal does not yet represent the measurement. This becomes a problem if the measured quantity changes quickly, because the filtered value may then have

a large error. Reducing RC reduces the lag, but the noise is then filtered less effectively, so there is a compromise.

The effect of RC on the response of the filter alone (with no noise) is shown in Figure 3 for a unit step input (see Appendix B for the MATLAB and Simulink files). As RC increases, the response becomes slower. The smallest value, $RC = 0.5$, gives the shortest lag, but it would filter a noisy measurement least effectively. This compromise is explored further in the Discrete-Time Modelling Learning Guide.

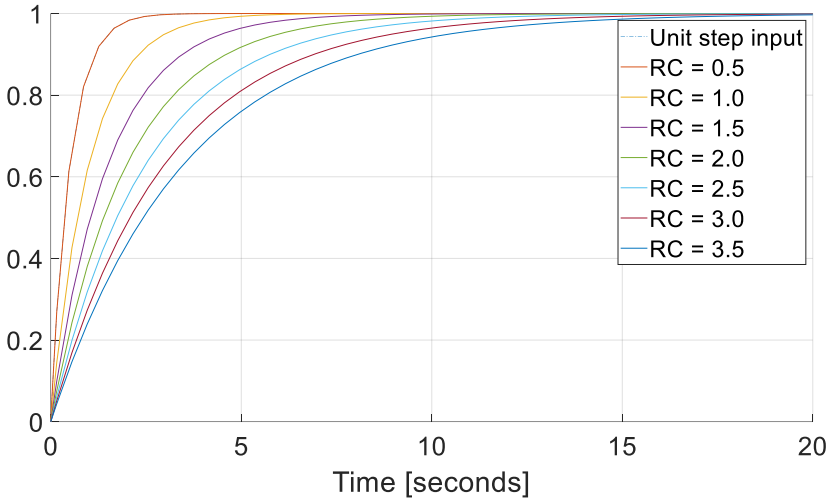


Figure 3: *Effect of RC on the Response of the RC Low-Pass Filter to a Unit Step Input*

Water Tank Level Control

The water tank (process) was modelled in the Laplace Transforms Learning Guide by the first-order transfer function:

$$G_p(s) = \frac{H(s)}{F_{in}(s)} = \frac{1}{As + \frac{\rho g}{R}} \tag{3}$$

where $F_{in}(s)$ is the flow in (input), $H(s)$ is the liquid level (output), A is the cross-sectional area of the tank, ρ is the liquid density, g is the acceleration due to gravity and R is the resistance of the outlet restriction. Since ρ and g are constants, changing $\rho g/R$ in practice means changing R.

Equation (3) is used to investigate the effect of A and $\rho g/R$ on the response to a unit step input (the MATLAB code is given in Appendix C):

- In Figure 4, $\rho g/R = 4$ and A is varied from 1 to 6. The final value is the same in every case (0.25 m), but increasing A makes the tank take longer to reach steady state.
- In Figure 5, $A = 2$ and $\rho g/R$ is varied from 3 to 8. Increasing $\rho g/R$ (that is, decreasing the restriction R, so that water flows out more easily) gives a lower final level, but the tank also reaches steady state more quickly.

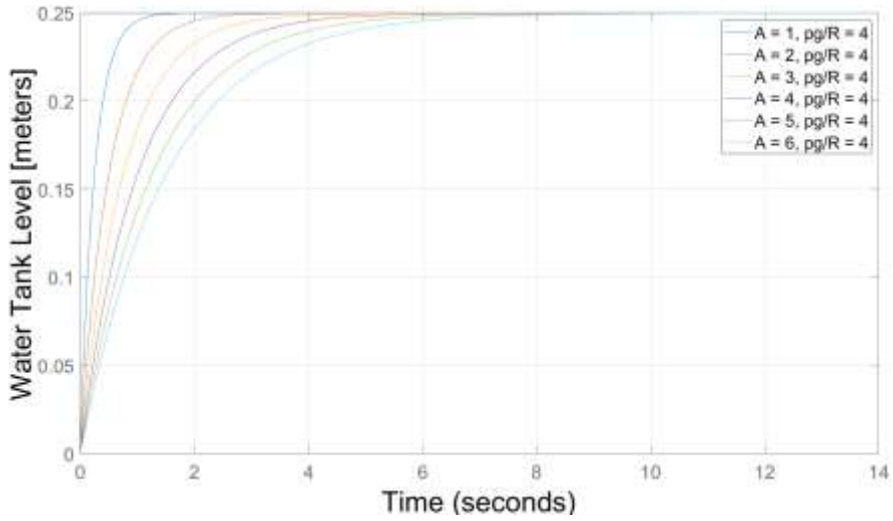


Figure 4: *Water Tank Response when the Area A is Varied, with $\rho g/R = 4$*

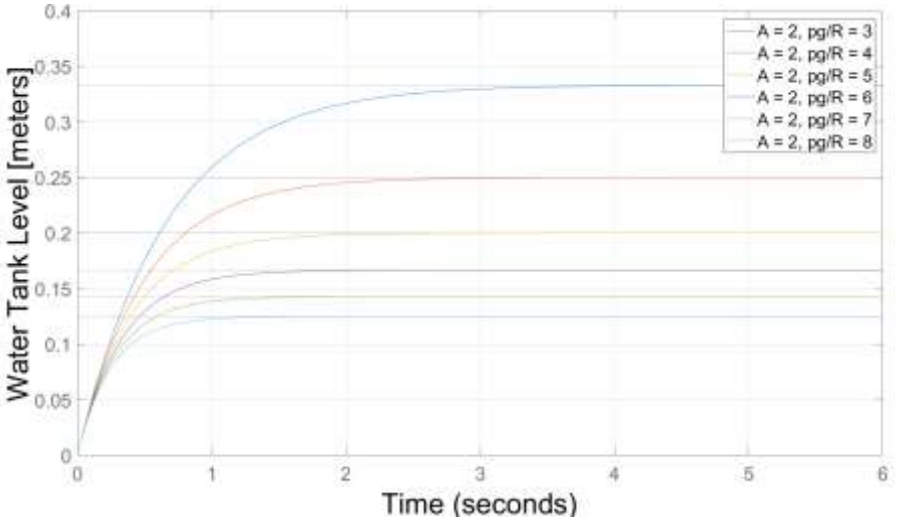


Figure 5: *Water Tank Response when $\rho g/R$ is Varied, with $A = 2$*

From the Final Value Theorem Learning Guide, the output of the closed-loop water tank level control system with a proportional control gain K_p, an ideal actuator and an ideal sensor is:

$$Y(s) = \left[\frac{K_p}{\left(As + \frac{\rho g}{R}\right) + K_p}\right]\frac{1}{s} \tag{4}$$

With $A = 1$ and $\rho g/R = 2$, this becomes:

$$Y(s) = G_{CL}(s)\,R(s) = \left[\frac{K_p}{(s+2) + K_p}\right]\frac{1}{s} \tag{5}$$

The time solution could be found using the inverse Laplace transform, but repeating the hand calculation for every change to the controller is time-consuming. Instead, Equation (5) has been simulated using MATLAB and Simulink for K_p from 5 to 35 in steps of 5, with a unit step reference (see

Appendix D). The results are shown in Figure 6. The final values for $K_p = 10$ and $K_p = 25$ match those found using the final value theorem:

$$K_p = 10: \quad y_{ss} = \frac{10}{12} = 0.833, \qquad e_{ss} = 16.7\%$$

$$K_p = 25: \quad y_{ss} = \frac{25}{27} = 0.926, \qquad e_{ss} = 7.4\%$$

As K_p increases, the final value increases and the SSE decreases. Figure 6 also shows that the response becomes faster as K_p increases, which the standard form in Section 4.0 explains.

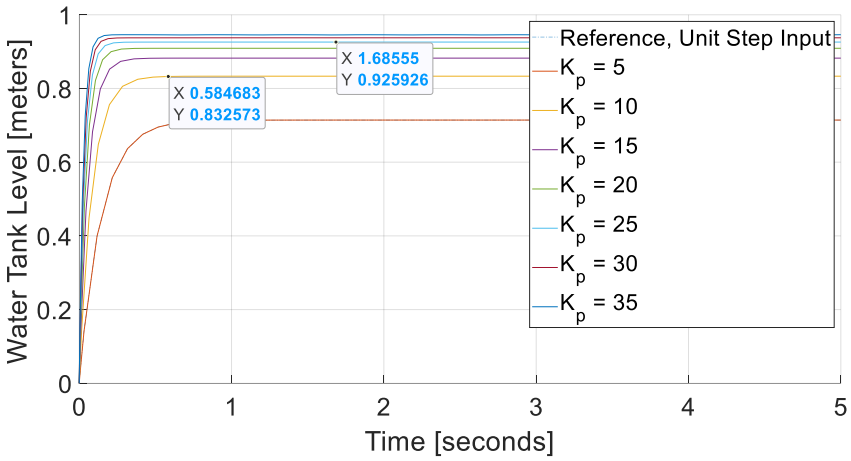


Figure 6: *Water Tank Level for a Range of Proportional Control Gain K_p Values*

4.0 The Standard Form

The standard form of a first-order transfer function is:

$$G(s) = \frac{K}{\tau s + 1} \tag{6}$$

where K is the system gain and τ is the time constant, in seconds. To put a first-order transfer function into standard form, divide the numerator and denominator by the constant term in the denominator, so that this term becomes 1. The numerator is then the system gain, K, and the coefficient of s is the time constant, τ. Equation (6) can represent either an open-loop system, $G(s)$, or a closed-loop control system, $G_{CL}(s)$.

The two parameters provide the following information:

- **System gain, K:** the final value of the output is the gain multiplied by the size of the step input:

$$y_{ss} = K\, u_{ss} \tag{7}$$

where u_{ss} is the magnitude of the step input (or reference). For a unit step, $y_{ss} = K$.

- **Time constant, τ:** for a step input, the time solution of Equation (6) is $y(t) = y_{ss}\left(1 - e^{-t/\tau}\right)$. After one time constant, the output has reached 63.2% of its final value, and after about five time constants it has effectively reached steady state.

Table 2 and Figure 7 show how far the output has risen after each multiple of τ. These values hold for every first-order system, which is why the standard form makes sketching a response so quick. As a rule of thumb, the output is taken to be in steady state after 5τ.

Table 2: *Response of a First-Order System to a Step Input*

Time	Output as a fraction of y_{ss}	Output as a percentage of y_{ss}
0	0	0%
τ	0.632	63.2%
2τ	0.865	86.5%
3τ	0.950	95.0%
4τ	0.982	98.2%
5τ	0.993	99.3%

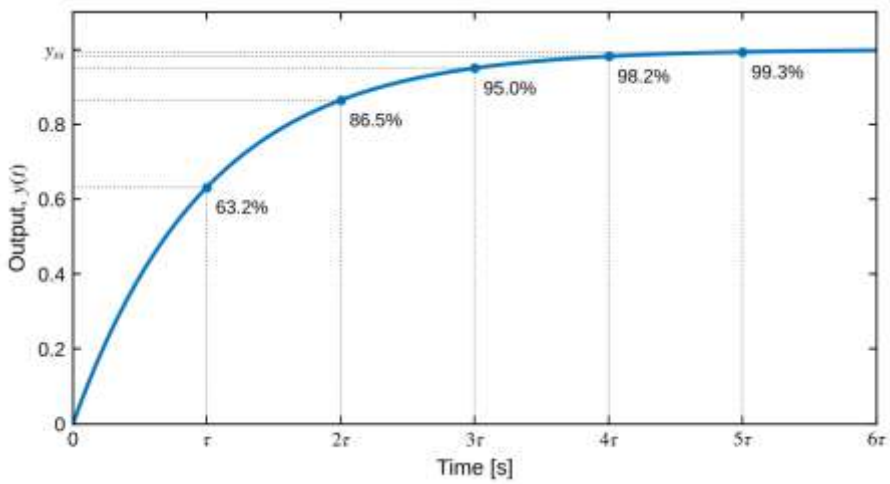


Figure 7: *Response of a First-Order System to a Step Input, Relating the Time Constant to the Output*

Example 1: Open-loop first-order system

A first-order open-loop system is modelled by the transfer function:

$$G(s) = \frac{5}{s+5}$$

Convert the transfer function into standard form, and sketch the output when a unit step input is applied.

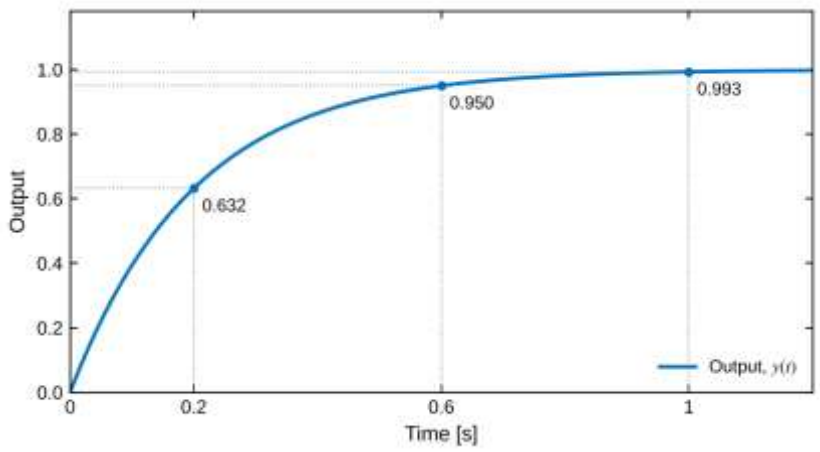
Solution:

Dividing the numerator and denominator by 5, so that the constant term in the denominator becomes 1, gives:

$$G(s) = \frac{1}{0.2s+1}$$

Comparing with Equation (6), the time constant is $\tau = 0.2$ s and the system gain is $K = 1$. For a unit step input, $y_{ss} = K = 1$, and Table 2 gives:

Time [s]	Output
0	0
$\tau = 0.2$	0.632
$3\tau = 0.6$	0.950
$5\tau = 1.0$	0.993



Example 2: Closed-loop first-order system

A closed-loop control system with a unit step reference is modelled by the transfer function:

$$G_{CL}(s) = \frac{2}{s+4}$$

Convert the transfer function into standard form, determine the steady-state error (SSE), and sketch the output and the reference.

Solution:

Dividing the numerator and denominator by 4 gives:

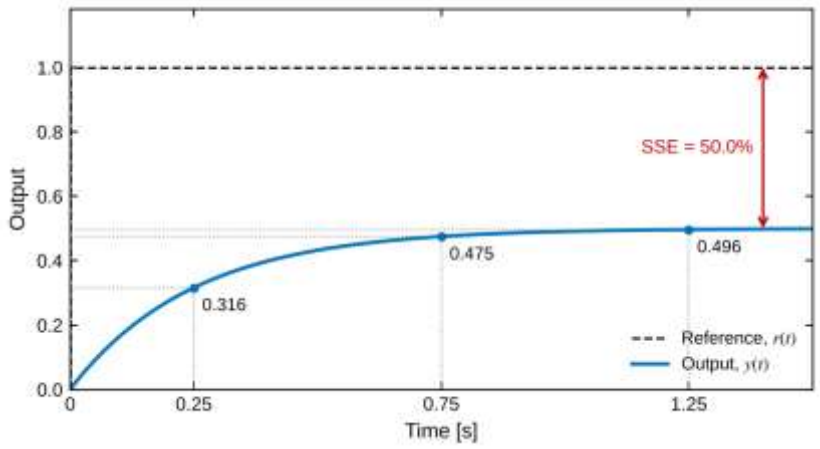
$$G_{CL}(s) = \frac{0.5}{0.25s + 1}$$

So $\tau = 0.25$ s and $K = 0.5$. For a unit step reference, $y_{ss} = 0.5$, and Table 2 gives:

Time [s]	Output
0	0
$\tau = 0.25$	0.632 × 0.5 = 0.316
$3\tau = 0.75$	0.950 × 0.5 = 0.475
$5\tau = 1.25$	0.993 × 0.5 = 0.497

The SSE is:

$$e_{ss}\,(\%) = \frac{1 - 0.5}{1} \times 100 = 50\%$$



Example 3: RC low-pass filter

The RC low-pass filter of Equation (2) has $R = 1000\ \Omega$ and $C = 1000\ \mu F$, and a unit step input of 1 V is applied.

Convert the transfer function into standard form, comment on the effect of R and C, and sketch the output voltage.

Solution:

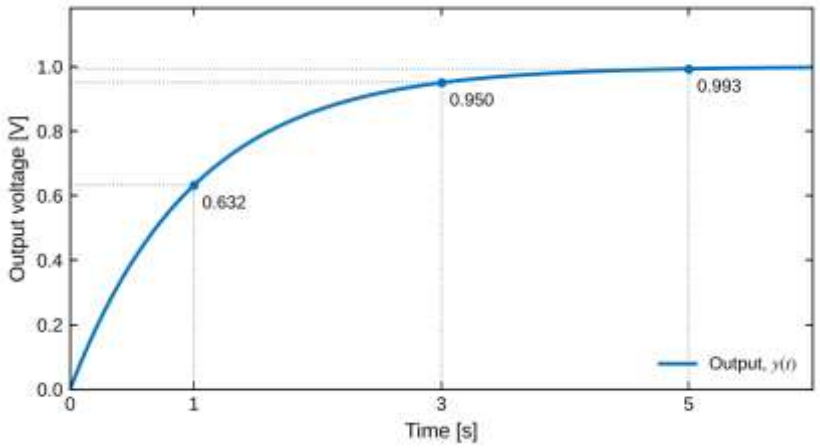
With $RC = 1000 \times 1000 \times 10^{-6} = 1$ s, the transfer function is:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s + 1}$$

This is already in standard form, with $K = 1$ and $\tau = RC = 1$ s. Since $\tau = RC$, a larger resistance or a larger capacitance gives a larger time constant, and

so a slower response. For a 1 V step, $y_{ss} = 1$ V, and Table 2 gives:

Time [s]	Output [V]
0	0
$\tau = 1$	0.632
$3\tau = 3$	0.950
$5\tau = 5$	0.993



This is the same response as the $RC = 1.0$ curve in Figure 3.

Example 4: Water tank level control

The closed-loop water tank level control system of Equation (5) has $K_p = 10$ and a unit step reference.

Convert the closed-loop transfer function into standard form, determine the steady-state error (SSE), and sketch the output and the reference.

Solution:

With $K_p = 10$, the closed-loop transfer function is:

$$G_{CL}(s) = \frac{10}{(s+2)+10} = \frac{10}{s+12}$$

Dividing the numerator and denominator by 12 gives:

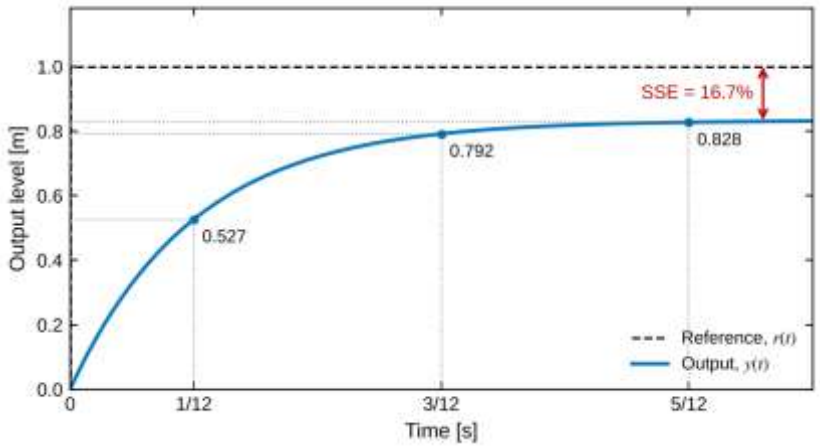
$$G_{CL}(s) = \frac{10/12}{(1/12)s+1}$$

So $\tau = 1/12$ s and $K = 10/12 = 0.833$. For a unit step reference, $y_{ss} = 0.833$ m, and Table 2 gives:

Time [s]	Output [m]
0	0
$\tau = 1/12$	0.632 × 0.833 = 0.527
$3\tau = 3/12$	0.950 × 0.833 = 0.792
$5\tau = 5/12$	0.993 × 0.833 = 0.828

The SSE is:

$$e_{ss}\,(\%) = \frac{1 - 10/12}{1} \times 100 = 16.7\%$$



This agrees with the $K_p = 10$ curve in Figure 6 and with the final value theorem. In general (see Appendix E), the closed-loop time constant is $\tau = A/(\rho g/R + K_p)$, so increasing K_p also makes the closed-loop response faster.

5.0 Exercises

Exercise 1: consider the following first-order transfer functions, each subject to a unit step input:

a) $G(s) = \frac{1}{s+1}$

b) $G(s) = \frac{2}{3s+6}$

c) $G(s) = \frac{3}{s+3}$

d) $G(s) = \frac{4}{2s+2}$

For each transfer function:

i. determine the time constant, τ, and the system gain, K

ii. sketch the output, labelling the final value and the outputs at τ, 3τ and 5τ

A step-by-step video solution for Questions a to d can be found on YouTube. Answers are given in Appendix F.

Exercise 2: an RC low-pass filter, $H(s) = 1/(RCs + 1)$, has $R = 2000\ \Omega$ and $C = 900\ \mu F$. Undertake tasks i and ii above for a unit step input.

Exercise 3: the closed-loop water tank level control system in Figure 8 has a proportional control gain $K_p = 15$, an ideal actuator, an ideal sensor and a unit step reference, with the process modelled by $G_p(s) = 1/(2s + 3)$. Undertake tasks i and ii above for the closed-loop control system, and also label the reference and the steady-state error (SSE).

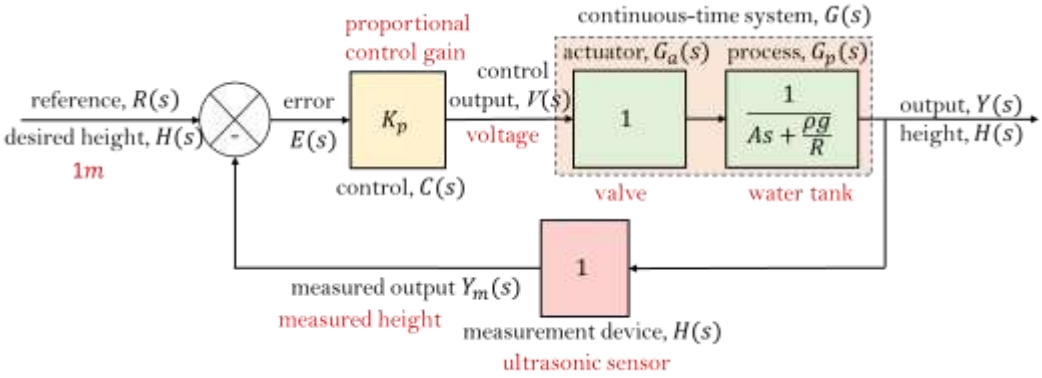


Figure 8: *Closed-Loop Control System for the Water Tank Level Control*

Fully worked solutions to Exercises 2 and 3 are given in Appendix F.

6.0 Summary

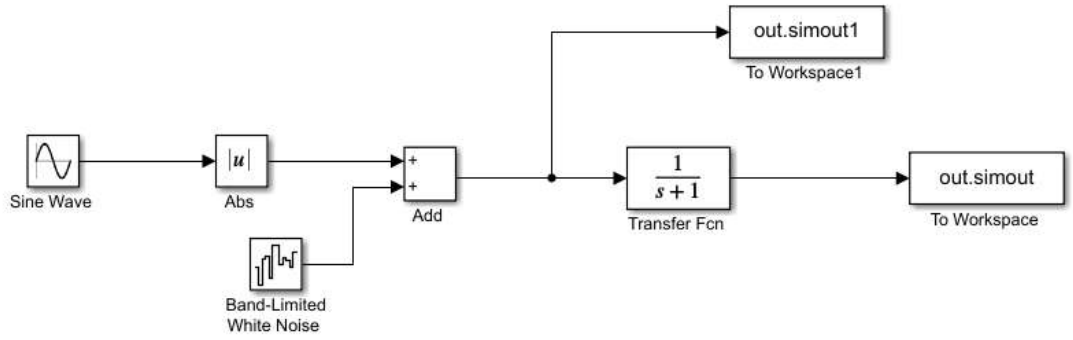
This Learning Guide considered the response of first-order transfer functions to step inputs. Using an RC low-pass filter and a water tank, MATLAB and Simulink were used to show how the system parameters and the proportional control gain, K_p, affect both the transient part of the response and the final value.

The standard form, $G(s) = K/(\tau s + 1)$, was then introduced. The system gain, K, multiplied by the size of the step gives the final value, and the time constant, τ, describes how quickly the output rises: 63.2% of the final value after τ, 95.0% after 3τ and 99.3% after 5τ. The standard form was applied to the earlier practical examples, confirming the simulated results and the steady-state error found using the final value theorem. For the closed-loop water tank, increasing K_p both reduced the steady-state error and reduced the time constant. The key progression introduced in this guide can be summarised as:

transfer function → standard form → K and τ → sketch of the response

Appendix A: Measurement Noise Filtered by an RC Low-Pass Filter

Simulink diagram:



MATLAB Script: simulating the noisy and filtered signals

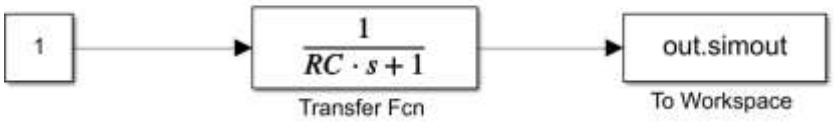
```
clear; clc; close all;

%% Simulation
sim('noise')

%% Graphical output
plot(ans.tout, ans.simout, ans.tout, ans.simout1)
set(gca, 'FontSize', 30)
xlabel('Time [s]', 'FontSize', 30);
legend('filtered signal', 'measured signal');
```

Appendix B: Effect of RC on the RC Low-Pass Filter

Simulink diagram:



MATLAB Script: simulating the RC low-pass filter for a range of RC values

```
% MATLAB version 2021a used to create this script
clear; clc; close all;

%% Graphical properties
figure; hold on; grid on;
xlabel('Time [s]');
set(gca, 'FontSize', 30);
idoitonce = 1;

%% Simulate a range of RC values
for RC = 0.5:0.5:3.5
    t = 20;       % simulation time
    sim('RC_low_pass_filter')
```

```
    if idoitonce  % plot the step input once
        plot(ans.tout, ans.simout(:,1), '-.');
        idoitonce = 0;
    end
    plot(ans.tout, ans.simout(:,1));
end
legend('Unit step input', 'RC = 0.5', 'RC = 1.0', ...
       'RC = 1.5', ...
       'RC = 2.0', 'RC = 2.5', 'RC = 3.0', 'RC = 3.5')
```

Appendix C: Water Tank System

MATLAB Script 1: effect of the tank area, A

```
clear; clc; close all;

for A = 1:1:6            % range of cross-sectional areas
    B = 4;               % rho*g/R
    sys = tf(1, [A B]);  % transfer function
    step(sys); hold all  % unit step response
end
set(gca, 'FontSize', 30)
xlabel('Time', 'FontSize', 30);
ylabel('Water Tank Level [m]', 'FontSize', 30);
grid on;
legend('A = 1', 'A = 2', 'A = 3', 'A = 4', 'A = 5', 'A = 6')
```

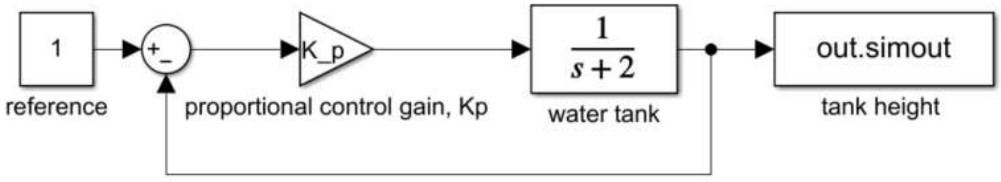
MATLAB Script 2: effect of $\rho g/R$

```
clear; clc; close all;

for B = 3:1:8            % range of rho*g/R values
    A = 2;               % cross-sectional area
    sys = tf(1, [A B]);  % transfer function
    step(sys); hold all  % unit step response
end
set(gca, 'FontSize', 30)
xlabel('Time', 'FontSize', 30);
ylabel('Water Tank Level [m]', 'FontSize', 30);
grid on;
legend('pg/R = 3', 'pg/R = 4', 'pg/R = 5', ...
       'pg/R = 6', 'pg/R = 7', 'pg/R = 8')
```

Appendix D: Water Tank Level Control

Simulink diagram:



MATLAB Script: simulating the closed-loop water tank for a range of K_p values

```
% MATLAB version 2021a used to create this script
clear; clc; close all;

%% Graphical properties
figure; hold on; grid on;
xlabel('Time [s]');
ylabel('Water Tank Level [m]');
set(gca, 'FontSize', 30);
idoitonce = 1;

%% Simulate a range of proportional control gains
for K_p = 5:5:35
    t = 5;        % simulation time
    sim('closed_loop_control_system')
    if idoitonce  % plot the reference once
        plot(ans.tout, ans.simout(:,1), '-.');
        idoitonce = 0;
    end
    plot(ans.tout, ans.simout(:,1));
end
legend('Reference', 'K_p = 5', 'K_p = 10', 'K_p = 15', ...
       'K_p = 20', ...
       'K_p = 25', 'K_p = 30', 'K_p = 35')
```

Appendix E: Water Tank Level Control in General Form

The open-loop transfer function of the water tank, Equation (3), is put into standard form by dividing the numerator and denominator by $\rho g/R$:

$$G_p(s) = \frac{1}{As + \frac{\rho g}{R}} = \frac{\frac{R}{\rho g}}{\frac{AR}{\rho g}s + 1}$$

so the system gain is $K = R/(\rho g)$ and the time constant is $\tau = AR/(\rho g)$. These explain Figures 4 and 5: A changes only the time constant, whereas $\rho g/R$ changes both the gain and the time constant.

The closed-loop control system with a proportional control gain K_p is:

$$G_{CL}(s) = \frac{K_p}{\left(As + \frac{\rho g}{R}\right) + K_p} = \frac{K_p}{As + \left(\frac{\rho g}{R} + K_p\right)}$$

Dividing the numerator and denominator by $(\rho g/R + K_p)$ gives the standard form:

$$G_{CL}(s) = \frac{\frac{K_p}{\rho g/R + K_p}}{\frac{A}{\rho g/R + K_p}\, s + 1}$$

so the closed-loop system gain is $K = K_p/(\rho g/R + K_p)$ and the closed-loop time constant is $\tau = A/(\rho g/R + K_p)$. Increasing K_p moves K closer to 1 (less SSE) and reduces τ (a faster response), as seen in Figure 6.

Appendix F: Solutions to Exercises

Exercise 1 (answers):

Table 3: *Answers to Exercise 1*

Question	Standard form	K	τ [s]
a	$\frac{1}{s+1}$	1	1
b	$\frac{1/3}{0.5s+1}$	1/3	0.5
c	$\frac{1}{(1/3)s+1}$	1	1/3
d	$\frac{2}{s+1}$	2	1

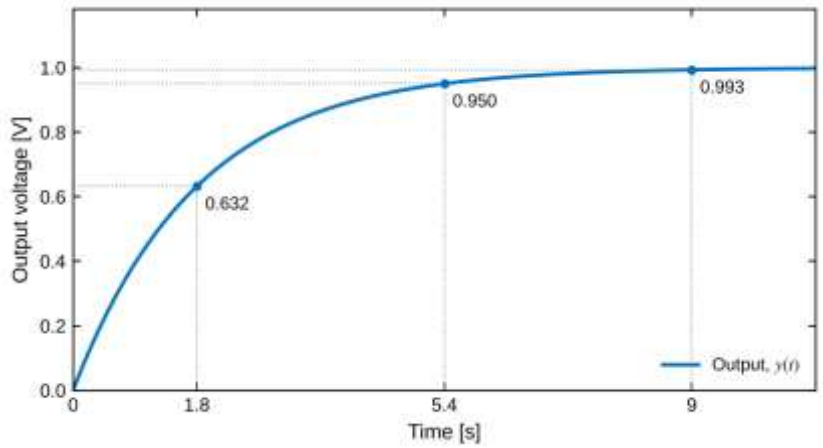
Exercise 2: RC low-pass filter

With $RC = 2000 \times 900 \times 10^{-6} = 1.8$ s, the transfer function is:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1.8s + 1}$$

This is already in standard form, with $K = 1$ and $\tau = 1.8$ s. For a unit step input, $y_{ss} = 1$, and Table 2 gives:

Time [s]	Output
0	0
$\tau = 1.8$	0.632
$3\tau = 5.4$	0.950
$5\tau = 9.0$	0.993



Exercise 3: Water tank level control

With $N(s) = 1$, $D(s) = 2s + 3$ and $K_p = 15$, the closed-loop transfer function is:

$$G_{CL}(s) = \frac{K_p\, N(s)}{D(s) + K_p\, N(s)} = \frac{15}{2s + 3 + 15} = \frac{15}{2s + 18}$$

Dividing the numerator and denominator by 18 gives the standard form:

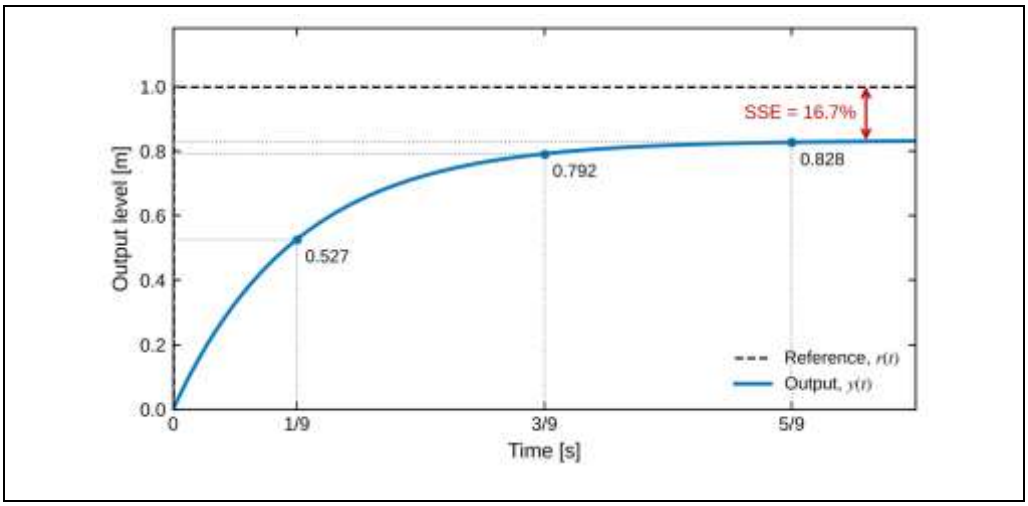
$$G_{CL}(s) = \frac{15/18}{(2/18)s + 1} = \frac{5/6}{(1/9)s + 1}$$

So $K = 5/6 = 0.833$ and $\tau = 1/9$ s. For a unit step reference, $y_{ss} = 0.833$, and Table 2 gives:

Time [s]	Output
0	0
$\tau = 1/9$	0.632 × 0.833 = 0.527
$3\tau = 3/9$	0.950 × 0.833 = 0.792
$5\tau = 5/9$	0.993 × 0.833 = 0.828

The SSE is:

$$e_{ss}\,(\%) = \frac{1 - 5/6}{1} \times 100 = 16.7\%$$



Recommended Citation

If you have used this learning guide in your teaching, learning or research, please cite it using the reference below, adapting the format as required by your preferred referencing style.

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