Final Value Theorem

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1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the Final Value Theorem Learning Guide*

ID	Learning Outcome	Achieved
1.	Explain the purpose of the final value theorem and use it to determine the steady-state output of a system without undertaking the inverse Laplace transform.	☐
2.	State the conditions under which the final value theorem gives a valid result.	☐
3.	Calculate the steady-state error (SSE) of a closed-loop control system by comparing the final value of the output with the reference.	☐
4.	Identify the Type of a system from its transfer function, and explain why Type 0 and Type 1 systems with proportional control give non-zero and zero SSE respectively for a step reference.	☐
5.	Explain how increasing the proportional control gain affects the SSE of a Type 0 system, using practical examples such as DC motor angle control and water tank level control.	☐

2.0 Introduction

In the Inverse Laplace Transform Learning Guide, the time solution of a transfer function was determined when a known input was applied. The time solution describes the complete dynamic behaviour of the system, including its final value once the transients have decayed and the system is in steady state. For a closed-loop control system, this final value can be compared with the reference, and the difference between the two is known as the steady-state error (SSE). Minimising the SSE is one of the fundamental aims of a closed-loop control system.

Obtaining the full-time solution requires several steps, including partial fractions and the table of transforms. Very often, however, only the final value is of interest. An important question therefore arises:

Can the final value of a system output be found without obtaining the full time solution?

The answer is the final value theorem. It gives the final value of the output directly from its Laplace transform, using a single limit. If the transfer function represents a closed-loop control system, the final value can then be compared with the reference to determine the SSE. This Learning Guide also introduces the Type of a system, which explains why some closed-loop control systems reach their reference exactly while others do not.

Why use the Final Value Theorem?

The final value theorem is useful because:

i. It gives the final value of the output of an open-loop system or a closed-loop control system, that is, the value of the output once the transients have decayed.
ii. Once the final value is known, the SSE of a closed-loop control system can be determined as the difference between the reference and the final value of the output.
iii. It is much quicker than undertaking the inverse Laplace transform, because no partial fractions or table look-ups are required.
iv. For a first- or second-order transfer function, the final value is directly related to the system gain, K, which is studied further in later Learning Guides.

3.0 Final Value of a System Output

In the Block Diagram Algebra Learning Guide, a closed-loop control system containing a proportional control gain, K_p, was introduced, as shown in Figure 1. With an ideal sensor, the output of this closed-loop control system is:

$$Y(s) = \left[\frac{K_p\, N(s)}{D(s) + K_p\, N(s)}\right] R(s) \tag{1}$$

where $G(s) = G_a(s)\,G_p(s) = N(s)/D(s)$ is the open-loop transfer function. Equation (1) and Figure 1 are used in this section to demonstrate the key properties of a system response. In particular, when the final value of the output is compared with the reference, the difference is the SSE.

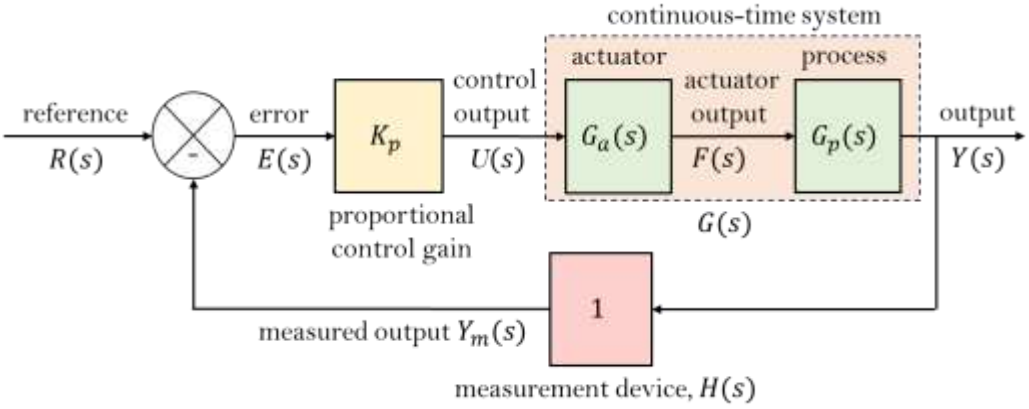


Figure 1: *Closed-Loop Control System Containing a Proportional Control Gain, K_p*

DC Motor Angle Control

The closed-loop control system for the DC motor angle control is shown in Figure 2. From Equation (1), its output is:

$$Y(s) = \left[\frac{113.7K_p}{s^2 + 3.398s + 113.7K_p}\right]\frac{180}{s} \tag{2}$$

The reference is a step of magnitude 180, $R(s) = 180/s$, which corresponds to an angle of 180° in the time domain. Equation (2) has been simulated using MATLAB and Simulink (the MATLAB code and Simulink diagram are given in Appendix A), and the result is shown in Figure 3. The output oscillates at first, but its final value is 180°. The closed-loop control system therefore achieves the reference, and the SSE is zero. As shown in Section 4.0, this is because the DC motor is a Type 1 system, and the result does not depend on the value of K_p.

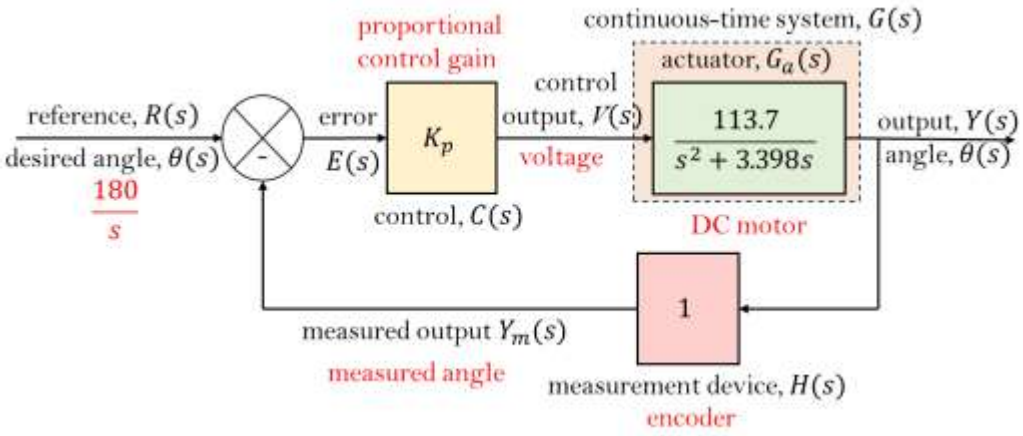


Figure 2: *Closed-Loop Control System for the DC Motor Angle Control Containing a Proportional Control Gain, K_p*

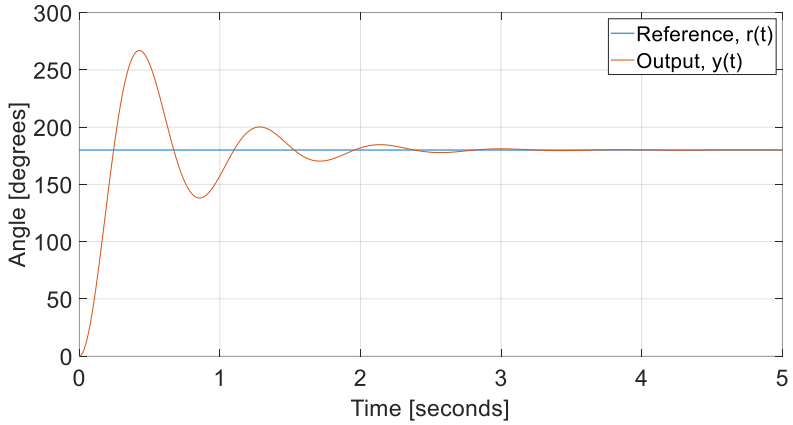


Figure 3: *Simulated Output of the Closed-Loop DC Motor Angle Control System*

Water Tank Level Control

The closed-loop control system for the water tank level control is shown in Figure 4. Assuming an ideal actuator, Equation (1) gives the output:

$$Y(s) = \left[\frac{K_p}{\left(As + \frac{\rho g}{R}\right) + K_p}\right]\frac{1}{s} \tag{3}$$

With $K_p = 1$, $A = 1$ and $\rho g/R = 2$, this becomes:

$$Y(s) = \frac{1}{s}\left[\frac{1}{(s+2)+1}\right] = \frac{1}{s}\left[\frac{1}{s+3}\right] \tag{4}$$

The reference is a unit step, $R(s) = 1/s$, which corresponds to a level of 1 m in the time domain. Equation (4) has been simulated using MATLAB and Simulink (see Appendix B), and the result is shown in Figure 5. In this case, the final value of the output does not reach the reference, and so there is a

non-zero SSE. As shown in Section 4.0, this is because the water tank is a Type 0 system.

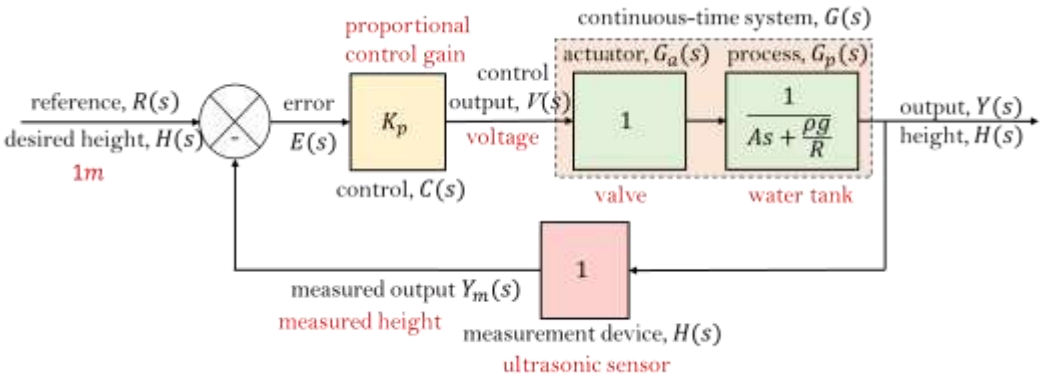


Figure 4: *Closed-Loop Control System for the Water Tank Level Control Containing a Proportional Control Gain, K_p*

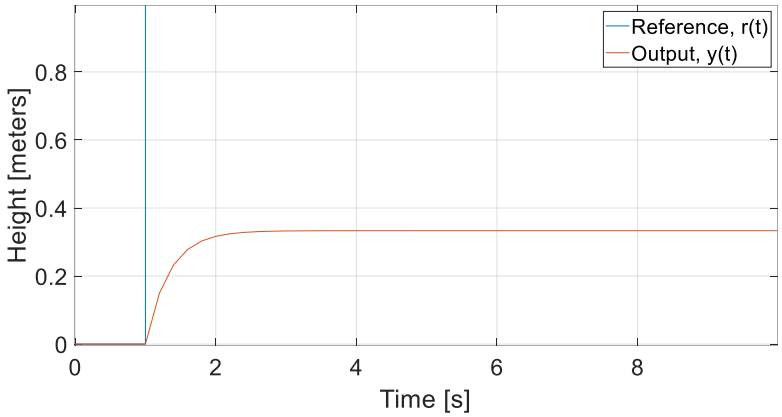


Figure 5: *Simulated Output of the Closed-Loop Water Tank Level Control System*

4.0 System Type

The **Type** of a system is the number of pure s terms that can be factorised out of the denominator of its open-loop transfer function, $G(s)$. Each such factor of s corresponds to a pole at $s = 0$, which acts as an integrator. The idea applies to systems of any order.

Type 1 Systems

A Type 1 system is one where a single factor of s can be taken out of the denominator. For example, the mass-damper system:

$$G(s) = \frac{1}{ms^2 + cs} = \frac{1}{s(ms + c)}$$

The DC motor in Figure 2 is also a Type 1 system:

$$G_a(s) = \frac{113.7}{s^2 + 3.398s} = \frac{113.7}{s(s + 3.398)}$$

An important property of a Type 1 system is that, within a stable closed-loop control system containing a proportional control gain, it has zero SSE for a step reference. The final value of the output is equal to the reference, as seen in Figure 3. This is because the integrator in the system continues to change the output for as long as any error remains.

Type 0 Systems

A Type 0 system is one where no factor of s can be taken out of the denominator. For example, the mass-spring-damper system:

$$G(s) = \frac{1}{ms^2 + cs + k}$$

The water tank in Figure 4 is also a Type 0 system:

$$G_p(s) = \frac{1}{As + \frac{\rho g}{R}}$$

Within a closed-loop control system containing a proportional control gain, a Type 0 system has a non-zero SSE for a step reference. The final value of the output does not reach the reference, as seen in Figure 5. Both properties are demonstrated using the final value theorem in Section 5.0.

5.0 The Final Value Theorem

The final value theorem is used to determine the final value of the output, as illustrated in Figure 6, without having to undertake the inverse Laplace transform. The final value can then be compared with the reference of a closed-loop control system to determine the SSE. The final value theorem is:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{5}$$

That is, the limit of $y(t)$ as t approaches infinity is equal to the limit of $sY(s)$ as s approaches zero. Recall that differentiation in the time domain corresponds to multiplying by s in the Laplace domain. In steady state, the output is no longer changing, and so setting $s \to 0$ removes the dynamic (transient) terms and leaves only the constant terms that determine the final value.

The final value theorem gives a valid result only if the output actually settles to a final value. This is the case when:

i. every pole of $sY(s)$ has a negative real part, that is, the closed-loop control system is stable, and
ii. $Y(s)$ has at most one pole at $s = 0$ (for example, the single pole contributed by a step input).

If these conditions are not met, for example if the system is unstable or oscillates indefinitely, the theorem still produces a number, but it has no physical meaning.

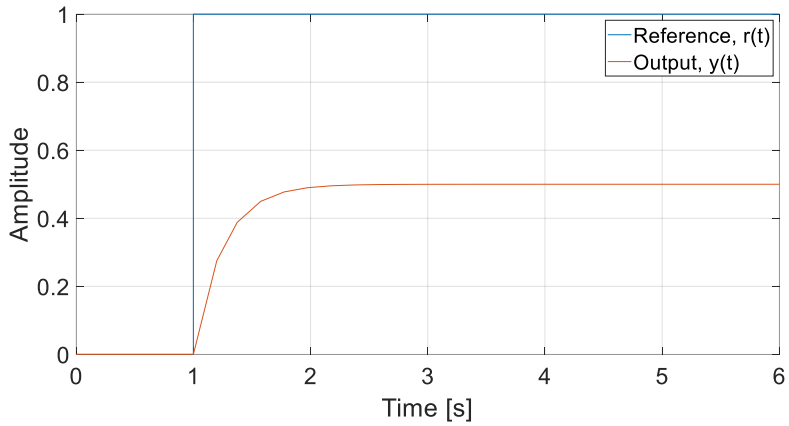


Figure 6: *Final Value of a System Output*

The SSE, denoted e_{ss}, is:

$$e_{ss} = r - y_{ss} \tag{6}$$

where r is the reference value and y_{ss} is the final value of the output in steady state. Dividing Equation (6) by r and multiplying by 100 expresses the SSE as a percentage of the reference:

$$e_{ss}\,(\%) = \frac{r - y_{ss}}{r} \times 100 \tag{7}$$

Example 1: Final value of a Type 0 system

A closed-loop control system has a proportional control gain $K_p = 1$, an ideal sensor and a unit step reference. The open-loop system is modelled by the transfer function:

$$G(s) = \frac{1}{s^2 + 3s + 1}$$

Determine the final value of the output and comment on the steady-state error (SSE).

Solution:

With $N(s) = 1$ and $D(s) = s^2 + 3s + 1$, Equation (1) gives:

$$Y(s) = \frac{1}{s}\left[\frac{(1)(1)}{(s^2 + 3s + 1) + (1)(1)}\right] = \frac{1}{s}\left[\frac{1}{s^2 + 3s + 2}\right]$$

Applying the final value theorem, Equation (5), the s terms cancel to give:

$$y_{ss} = \lim_{s \to 0} s\,\frac{1}{s}\left[\frac{1}{s^2 + 3s + 2}\right] = \frac{1}{2} = 0.5$$

The final value of the output is 0.5. From Equation (7), the SSE is:

$$e_{ss}\,(\%) = \frac{1-0.5}{1} \times 100 = 50\%$$

$G(s)$ is a Type 0 system, and so a non-zero SSE is expected.

Example 2: Effect of the proportional control gain on a Type 0 system

Following on from Example 1, the proportional control gain now takes values of $K_p = 1$, 10 and 25. Determine the final value of the output for each value and comment on the steady-state error (SSE).

Solution:

Keeping K_p as a variable, the output is:

$$Y(s) = \frac{1}{s}\left[\frac{K_p}{s^2 + 3s + 1 + K_p}\right]$$

Applying the final value theorem gives:

$$y_{ss} = \lim_{s\to 0} s\,\frac{1}{s}\left[\frac{K_p}{s^2 + 3s + 1 + K_p}\right] = \frac{K_p}{1 + K_p}$$

Substituting each value of K_p:

K_p	**Final value,** y_{ss}	**SSE,** e_{ss} **(%)**
1	0.50	50.0
10	0.91	9.1
25	0.96	3.8

For a Type 0 system, increasing K_p reduces the SSE but never removes it completely. As later Learning Guides will show, the reduction in SSE comes at the expense of a more oscillatory transient response.

Example 3: DC motor angle control

The closed-loop DC motor angle control system in Figure 2 has a proportional control gain $K_p = 0.5$ and a reference of 180°. The actuator is modelled by the transfer function:

$$G_a(s) = \frac{113.7}{s^2 + 3.398s}$$

Determine the final value of the output and comment on the steady-state error (SSE).

Solution:

With $N(s) = 113.7$, $D(s) = s^2 + 3.398s$ and $R(s) = 180/s$, Equation (1) gives:

$$Y(s) = \frac{180}{s}\left[\frac{(0.5)(113.7)}{(s^2 + 3.398s) + (0.5)(113.7)}\right] = \frac{180}{s}\left[\frac{56.85}{s^2 + 3.398s + 56.85}\right]$$

Applying the final value theorem gives:

$$y_{ss} = \lim_{s\to 0} s\,\frac{180}{s}\left[\frac{56.85}{s^2 + 3.398s + 56.85}\right] = 180 \times \frac{56.85}{56.85} = 180$$

The final value of the output is 180°, and so the SSE is:

$$e_{ss}\,(\%) = \frac{180 - 180}{180} \times 100 = 0\%$$

The closed-loop control system has no SSE, and the DC motor reaches the desired angle. As discussed in Section 4.0, this is expected because the DC motor is a Type 1 system. Note that K_p cancels in the final step, so the same result is obtained for any value of K_p that keeps the system stable.

Example 4: Water tank level control

A closed-loop water tank level control system has a proportional control gain of $K_p = 1$, 10 and 25, an ideal actuator, an ideal sensor and a reference of 1 m (a unit step). The process is modelled by the transfer function:

$$G_p(s) = \frac{1}{s + 2}$$

Determine the final value of the output for each value of K_p and comment on the steady-state error (SSE).

Solution:

With $N(s) = 1$ and $D(s) = s + 2$, and keeping K_p as a variable, Equation (1) gives:

$$Y(s) = \frac{1}{s}\left[\frac{K_p}{(s + 2) + K_p}\right] = \frac{1}{s}\left[\frac{K_p}{s + (2 + K_p)}\right]$$

Applying the final value theorem gives:

$$y_{ss} = \lim_{s\to 0} s\,\frac{1}{s}\left[\frac{K_p}{s + (2 + K_p)}\right] = \frac{K_p}{2 + K_p}$$

Substituting each value of K_p:

K_p	**Final value,** y_{ss} **(m)**	**SSE,** e_{ss} **(%)**
1	1/3 = 0.333	66.7
10	10/12 = 0.833	16.7
25	25/27 = 0.926	7.4

As in Example 2, increasing K_p reduces the SSE of this Type 0 system, but does not remove it.

6.0 Exercises

Exercise 1: consider the following closed-loop transfer functions, where a unit step input is used to model the reference:

a) $G_{CL}(s) = \frac{2}{s+4}$

b) $G_{CL}(s) = \frac{3}{s^2+6s+5}$

c) $G_{CL}(s) = \frac{4}{2s^2+2s+4}$

d) $G_{CL}(s) = \frac{7}{2s^2+3s+14}$

For each closed-loop transfer function, undertake the following tasks:

i. determine the final value of the output
ii. comment on the steady-state error (SSE)
iii. sketch the form of the output and compare it with the reference

A step-by-step video solution for Questions a to d can be found on YouTube.

Exercise 2: for the soldering robot displacement control system in Figure 7, undertake tasks i to iii above.

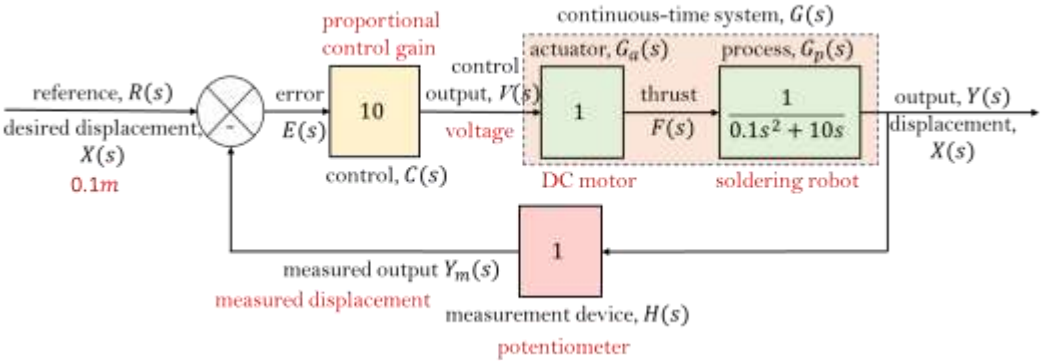


Figure 7: *Closed-Loop Control System for the Soldering Robot Displacement Control*

Exercise 3: for the pendulum angle control system in Figure 8, undertake tasks i to iii above.

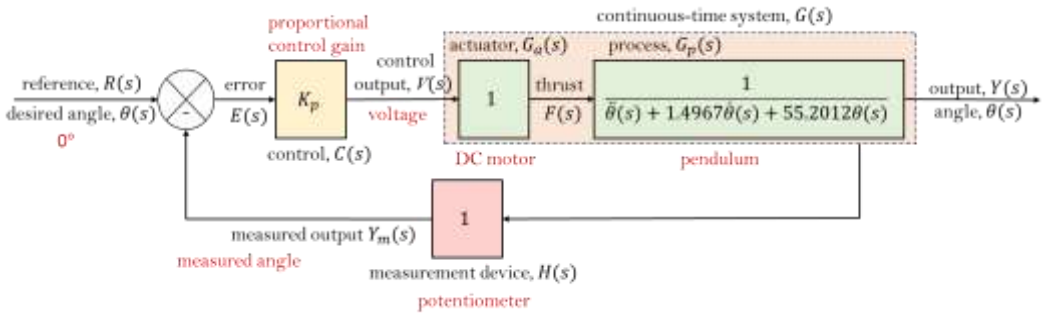


Figure 8: *Closed-Loop Control System for the Pendulum Angle Control*

7.0 Summary

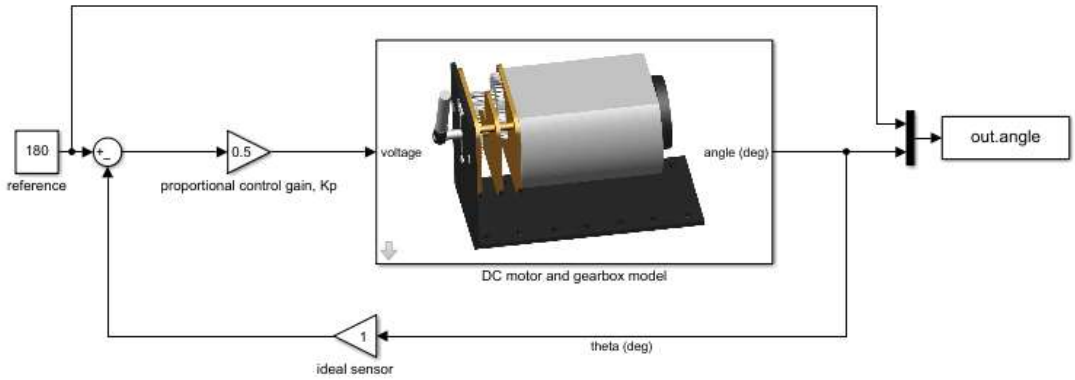
This Learning Guide introduced the final value theorem, which gives the final value of the output of an open-loop system or a closed-loop control system directly from its Laplace transform. Through examples, it was shown how the final value is compared with the reference to determine the steady-state error (SSE) of a closed-loop control system. The final value theorem is much quicker than obtaining the full time solution using the inverse Laplace transform. However, it gives no information about the transient part of the response, which is studied in the Learning Guides on the response of first- and second-order transfer functions. In practice, MATLAB and Simulink are widely used to obtain the complete response.

The Type of a system was also introduced. With a proportional control gain and a step reference, a Type 1 system has zero SSE, whereas a Type 0 system has a non-zero SSE, which reduces as K_p is increased. The key progression introduced in this guide can be summarised as:

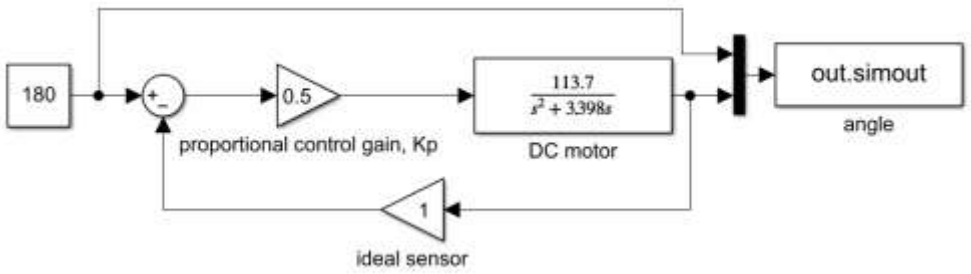
closed-loop output → final value theorem → final value → steady-state error

Appendix A: DC Motor Angle Control (MATLAB and Simulink)

Simulink diagram (high-fidelity model):



Simulink diagram (low-fidelity model):



MATLAB Script: simulating the DC motor angle control system and plotting the output

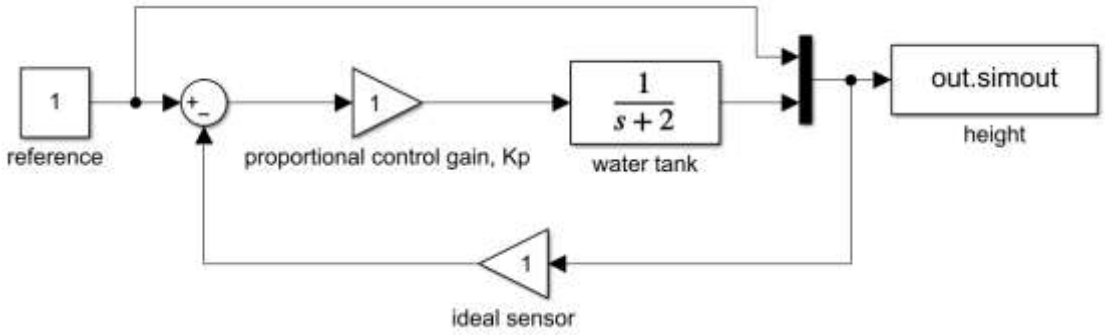
```
clear; clc; close all;

%% Simulink
sim('dc_motor')

plot(ans.tout, ans.angle)
set(gca, 'FontSize', 30);   % set all text to size 30
xlabel('Time [s]');         % x-axis label
ylabel('Angle [degrees]');  % y-axis label
legend('Reference, r(t)', 'Output, y(t)');
grid on
```

Appendix B: Water Tank Level Control (MATLAB and Simulink)

Simulink diagram:



MATLAB Script: simulating the water tank level control system and plotting the output

```
clear; clc; close all;

%% Simulink
sim('water_tank')

plot(ans.tout, ans.simout)
set(gca, 'FontSize', 30);  % set all text to size 30
xlabel('Time [s]');        % x-axis label
ylabel('Height [m]');      % y-axis label
legend('Reference, r(t)', 'Output, y(t)');
grid on
```

Recommended Citation

If you have used this learning guide in your teaching, learning or research, please cite it using the reference below, adapting the format as required by your preferred referencing style.

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