Discrete-Time Modelling

1.0 Learning Outcomes....1
2.0 Introduction....2
3.0 Sampled-Data Control Systems....2
4.0 The Zero-Order Hold....5
5.0 Discrete-Time Transfer Functions....8
6.0 Exercises....10
7.0 Summary....11
Appendix A: Solutions to Exercise 1....12
Appendix B: Answers to Exercise 2....15
Recommended Citation....15

1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the Discrete-Time Modelling Learning Guide*

ID	Learning Outcome	Achieved
1.	Explain the structure and operation of a sampled-data control system, including the roles of the sampling switch, ADC, DAC and zero-order hold (ZOH), and the effect of the sampling interval.	☐
2.	Explain the definition and purpose of the z-transform, the meaning of the backward time-shift operator and the relationship z = e^(sTs).	☐
3.	Use a table of transforms to convert functions between the time, Laplace and z domains.	☐
4.	Derive the transfer function of the ZOH and use it to discretise a continuous-time transfer function by partial fractions, the table of transforms and the ZOH numerator.	☐
5.	Verify a discretised transfer function using MATLAB.	☐

2.0 Introduction

The previous Learning Guides modelled control systems in continuous time, using the Laplace transform and transfer functions in s. In practice, however, almost every modern controller is a control algorithm running on a microcontroller or computer. The algorithm cannot see a continuous signal: it receives measurements at fixed instants, separated by a sampling interval, T_s, and it updates its output only at those instants. An important question therefore arises:

How can a system that combines a digital controller with a continuous-time process be modelled?

The answer is the sampled-data control system, which is modelled using the z-transform. This Learning Guide describes the components of a sampled-data control system, introduces the z-transform and its table of transforms, explains the zero-order hold (ZOH), and shows how a continuous-time transfer function, $G(s)$, is converted (discretised) into an equivalent discrete-time transfer function, $G(z)$.

The choice of sampling interval is central throughout. A shorter sampling interval gives the controller more up-to-date information and generally a better response, but it increases the computational load. A longer sampling interval reduces the computational load, but the controller reacts more slowly, and if the signal is sampled too slowly compared with how fast it changes (that is, if the Nyquist criterion is not satisfied), aliasing can occur and the sampled signal no longer represents the true signal.

Why are Discrete-Time Models Important?

Discrete-time models are important because:

i. Digital controllers operate on sampled signals, so their behaviour is described exactly by discrete-time models.
ii. Discrete-time models allow the effect of the sampling interval and the ZOH on the stability and response of a control system to be analysed before it is implemented.
iii. A control algorithm designed in discrete time can be implemented directly as a difference equation in code.
iv. Most digital signal processing, such as filtering, operates on discrete-time signals.

3.0 Sampled-Data Control Systems

A sampled-data control system combines continuous-time and discrete-time signals, as shown in Figure 1 for a cabin heating and cooling system. The process, actuator and sensor operate in continuous time, whereas the control algorithm operates on samples taken every T_s seconds. The components that link the two are:

- **Sampling switch (SS):** samples the continuous-time signal at each sampling instant to give a discrete-time sequence. For example, the

measured output $y(t)$ becomes the sampled output y_k, where k is the sample number.

- **Analogue-to-digital converter (ADC):** converts each sample of an analogue voltage into a digital value that the algorithm can process. In Figure 1, the sampling switch and ADC together produce the sampled reference r_k and sampled output y_k.
- **Control algorithm:** calculates the control output u_k from the sampled error $e_k = r_k - y_k$.
- **Digital-to-analogue converter (DAC) and zero-order hold (ZOH):** convert each value u_k back into a continuous signal, $u^*(t)$, by holding it constant until the next sample. This produces the "staircase" signal shown in Figure 1.

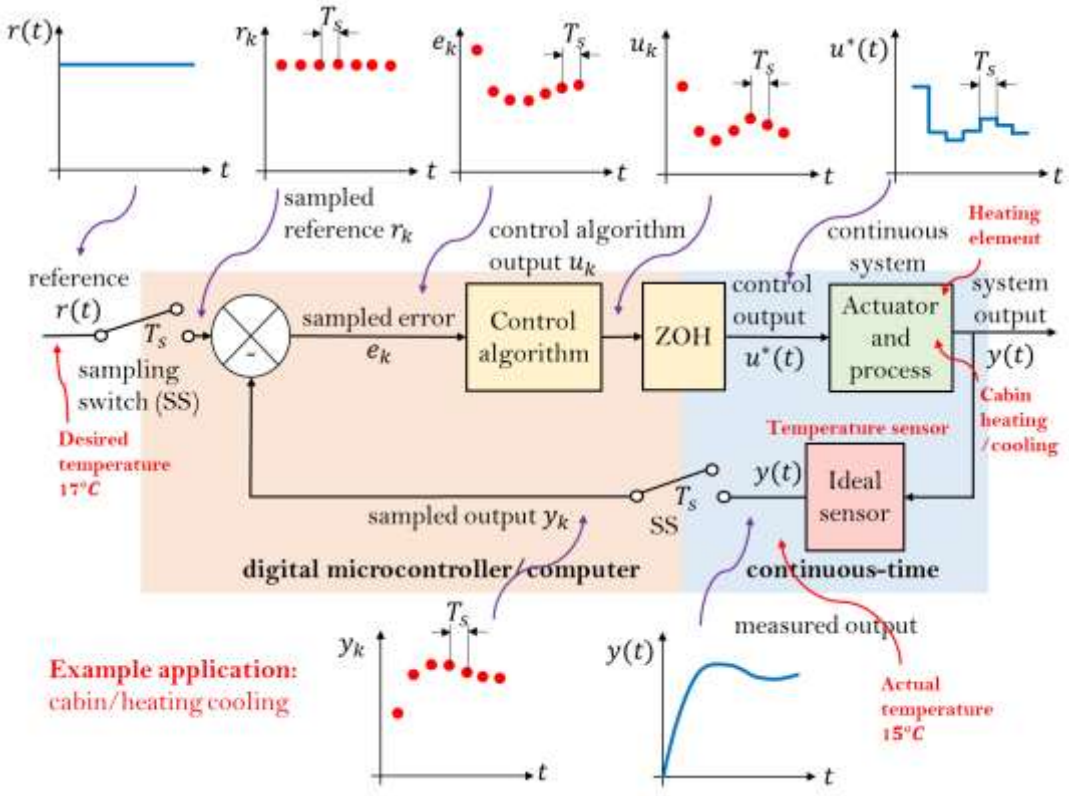


Figure 1: *A Sampled-Data Control System, Illustrated for Cabin Heating and Cooling*

The z-Transform

A continuous signal $f(t)$ that is sampled every T_s seconds gives the sequence of values $f_0, f_1, f_2, \ldots$, where $f_k = f(kT_s)$, as illustrated in Figure 2. The z-transform of this sequence is defined as:

$$\mathcal{Z}\{f(t)\} = F(z) = \sum_{k=0}^{\infty} f_k \, z^{-k} \tag{1}$$

where $k = 0,1,2, \ldots$ is the sample number. Each power of z^{-1} marks how many sampling intervals a sample is delayed by: multiplying a signal by z^{-1} delays it

by one sampling interval, T_s. For this reason, z^{-1} is known as the backward time-shift operator.

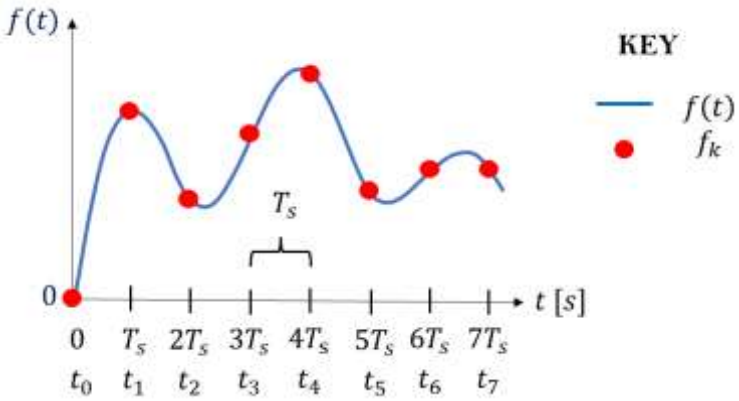


Figure 2: *A Continuous Signal and its Samples*

The z-transform plays the same role for discrete-time systems as the Laplace transform does for continuous-time systems. It turns difference equations into algebraic equations, which allows discrete-time transfer functions to be formed, stability to be assessed, and digital controllers and filters to be designed. Just as the inverse Laplace transform gives a time solution, the inverse z-transform gives the values of the output at the sampling instants. In practice, a digital controller can only use present and past samples, and so it is written in terms of the backward time-shift operator, z^{-1}, as later Learning Guides will show.

Table of Transforms

A table of transforms, Table 2, gives the z-transform of common functions alongside their Laplace transforms. It is used in the same way as the table of transforms for the Laplace transform: once a function of s has been broken into simple terms, each term can be converted to z by looking it up. The table is used in Sections 4.0 and 5.0 to discretise continuous-time transfer functions.

Table 2: *Table of Transforms Between the Time, Laplace and z Domains*

Row	$x(t)$	$X(s)$	$X(z)$
1	$\delta(t)$	1	1
2	$\delta(t-kT_s)$	$e^{-kT_s s}$	z^{-k}
3	1 (unit step)	$\frac{1}{s}$	$\frac{z}{z-1}$
4	t	$\frac{1}{s^2}$	$\frac{T_s z}{(z-1)^2}$
5	t^2	$\frac{2}{s^3}$	$\frac{T_s^2 z(z+1)}{(z-1)^3}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	$\frac{z}{z-e^{-\alpha T_s}}$
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$\frac{(1-e^{-\alpha T_s})z}{(z-1)(z-e^{-\alpha T_s})}$
8	$t\,e^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$	$\frac{T_s z\,e^{-\alpha T_s}}{(z-e^{-\alpha T_s})^2}$
9	$\frac{e^{-\beta t}-e^{-\alpha t}}{\alpha-\beta}$	$\frac{1}{(s+\alpha)(s+\beta)}$	$\frac{z\left(e^{-\beta T_s}-e^{-\alpha T_s}\right)}{(\alpha-\beta)(z-e^{-\alpha T_s})(z-e^{-\beta T_s})}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin(\omega T_s)}{z^2-2z\cos(\omega T_s)+1}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\frac{z\left(z-\cos(\omega T_s)\right)}{z^2-2z\cos(\omega T_s)+1}$
12	$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$\frac{z\,e^{-\alpha T_s}\sin(\omega T_s)}{z^2-2z\,e^{-\alpha T_s}\cos(\omega T_s)+e^{-2\alpha T_s}}$
13	$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$\frac{z^2-z\,e^{-\alpha T_s}\cos(\omega T_s)}{z^2-2z\,e^{-\alpha T_s}\cos(\omega T_s)+e^{-2\alpha T_s}}$

Two properties of the z-transform are also useful. For a sequence x_k with z-transform $X(z)$, a delay of one sample gives $\mathcal{Z}\{x_{k-1}\} = z^{-1}X(z)$ (assuming $x_{-1} = 0$), and an advance of one sample gives $\mathcal{Z}\{x_{k+1}\} = zX(z) - z\,x_0$. These are the discrete-time equivalents of the Laplace transform of a derivative.

4.0 The Zero-Order Hold

The zero-order hold (ZOH) converts the sequence of control outputs, u_k, into the continuous-time staircase signal, $u^*(t)$, that drives the actuator, as shown in Figure 3. It holds each value of u_k constant for one sampling interval, until the next value arrives. This is essential because the controller operates in discrete time, whereas the actuator and process operate in continuous time. The process repeats every T_s seconds, with each new u_k calculated from the latest sampled error.

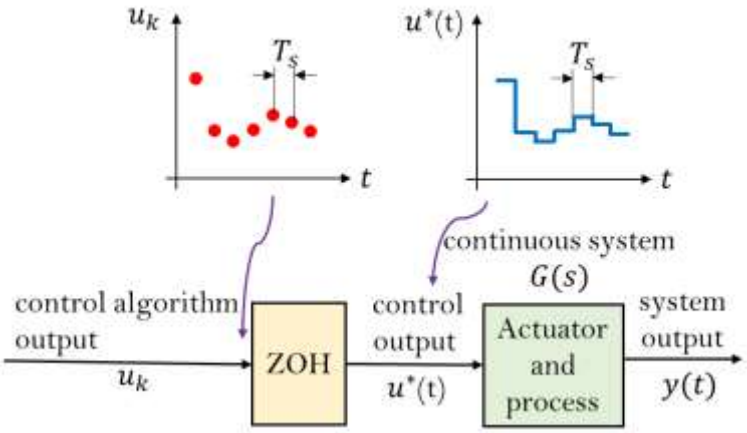


Figure 3: *Operation of the Zero-Order Hold (ZOH)*

Transfer Function of the ZOH

Each held value is a rectangular pulse of height u_k and width T_s, as illustrated in Figure 4 for the samples $u_0 = 5$, $u_1 = 3$ and $u_2 = 8$. A single pulse of unit height can be built from two steps: a unit step applied at the start of the sampling interval, and a unit step removed (subtracted) one sampling interval later. Using the table of transforms, a unit step is $1/s$, and a delay of T_s multiplies by e^{-sT_s}. The transfer function of the ZOH is therefore:

$$G_{ZOH}(s) = \frac{1}{s} - \frac{1}{s}e^{-sT_s} = \frac{1 - e^{-sT_s}}{s} \tag{2}$$

For the samples in Figure 4, the first pulse is a step of 5 applied at $t = 0$ and removed at $t = T_s$; the second is a step of 3 applied at $t = T_s$ and removed at $t = 2T_s$; and so on. Adding the pulses together gives the staircase signal $u^*(t)$:

$$U^*(s) = \frac{1 - e^{-sT_s}}{s}(5 + 3e^{-sT_s} + 8e^{-2sT_s})$$

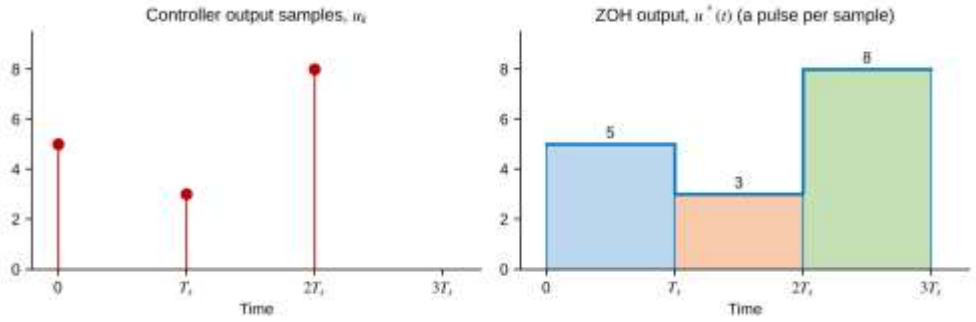


Figure 4: *The ZOH Output Built from One Pulse per Sample*

On average, the ZOH output lags the samples by half a sampling interval. This delay can destabilise a sampled-data control system if the sampling interval is too long compared with the speed of the system, which is covered in later Learning Guides.

The z-Form of the ZOH

Consider the continuous-time signal $f(t) = e^{-at}$, for $t \geq 0$, which has the Laplace transform:

$$F(s) = \frac{1}{s + a} \tag{3}$$

with a pole at $s = -a$. From Row 6 of Table 2, its z-transform is:

$$F(z) = \frac{z}{z - e^{-aT_s}} \tag{4}$$

with a pole at $z = e^{-aT_s}$. Comparing the two poles shows that the s-domain and the z-domain are related by:

$$z = e^{sT_s} \tag{5}$$

and so $z^{-1} = e^{-sT_s}$: a delay of one sampling interval. Replacing e^{-sT_s} with z^{-1} in the numerator of the ZOH, Equation (2), gives:

$$1 - e^{-sT_s} = 1 - z^{-1} = \frac{z - 1}{z} \tag{6}$$

The discrete-time transfer function of a continuous-time system $G(s)$ preceded by a ZOH, illustrated in Figure 5, is:

$$G(z) = \mathcal{Z}\{G_{ZOH}(s)\, G(s)\} = \mathcal{Z}\left\{\frac{1 - e^{-sT_s}}{s} G(s)\right\} \tag{7}$$

Since the numerator of the ZOH is simply $(z - 1)/z$ in the z-domain, from Equation (6), this becomes:

$$G(z) = \frac{z - 1}{z}\, \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \tag{8}$$

Equation (8) is the key result used to discretise a continuous-time transfer function. It describes the input–output behaviour of the ZOH and the continuous-time system exactly at the sampling instants.

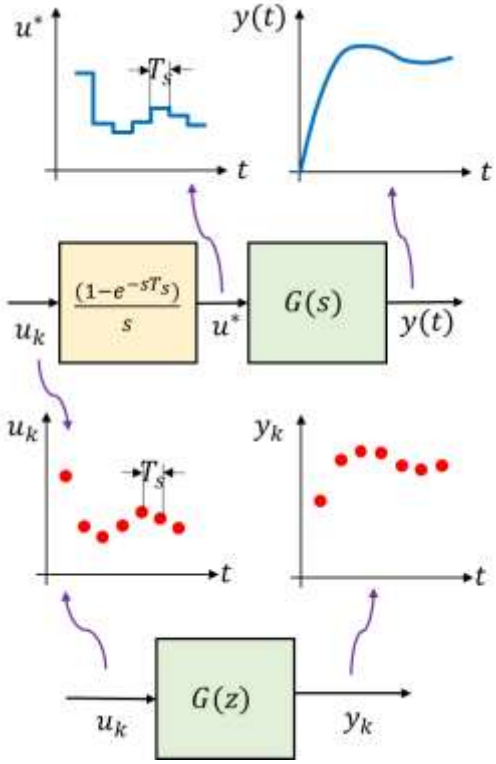


Figure 5: *Converting the ZOH and Continuous-Time System into a Discrete-Time Transfer Function*

5.0 Discrete-Time Transfer Functions

A discrete-time transfer function, $G(z)$, relates the sampled output of a system to its sampled input, in the same way that $G(s)$ relates the output to the input in continuous time. It is used to analyse the stability and response of a sampled-data control system, and to design and implement digital controllers.

Steps to Discretise a Transfer Function

Using Equation (8), a continuous-time transfer function is discretised in four steps:

1. **Partial fractions:** express $G(s)/s$ as a sum of partial fractions, so that each term appears in the table of transforms.
2. **Table of transforms:** convert each term from the s-domain to the z-domain using Table 2.
3. **Common denominator:** combine the terms over a common denominator to form a single transfer function.
4. **Multiply by the ZOH numerator:** multiply by $(z-1)/z$ and simplify to give $G(z)$.

Example 1: Discretising a second-order system

A sampled-data control system has a sampling interval $T_s = 1$ s, and the open-loop system is:

$$G(s) = \frac{12}{(s+2)(s+3)}$$

Using $G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$, show that the discrete-time transfer function is:

$$G(z) = \frac{1.3871z + 0.2561}{z^2 - 0.1851z + 0.0067}$$

You may assume that the partial fractions take the form $\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$.

Step 1: Partial fractions

$$\frac{G(s)}{s} = \frac{12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

Multiplying both sides by $s(s+2)(s+3)$ and expanding the brackets gives:

$$12 = A(s^2 + 5s + 6) + B(s^2 + 3s) + C(s^2 + 2s)$$

Equating the coefficients of each power of s gives:

$$s^0\text{: } 12 = 6A \qquad s^1\text{: } 0 = 5A + 3B + 2C \qquad s^2\text{: } 0 = A + B + C$$

From the s^0 equation, $A = 2$. Multiplying the s^2 equation by 2 and subtracting it from the s^1 equation gives $0 = 3A + B$, so $B = -6$. Substituting into the s^2 equation gives $0 = 2 - 6 + C$, so $C = 4$. Hence:

$$\frac{G(s)}{s} = \frac{2}{s} - \frac{6}{s+2} + \frac{4}{s+3}$$

Step 2: Table of transforms

Using Rows 3 and 6 of Table 2:

$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{2z}{z-1} - \frac{6z}{z-e^{-2T_s}} + \frac{4z}{z-e^{-3T_s}} = 2z\left(\frac{1}{z-1} - \frac{3}{z-e^{-2T_s}} + \frac{2}{z-e^{-3T_s}}\right)$$

Step 3: Common denominator

$$= 2z\left(\frac{(z-e^{-2T_s})(z-e^{-3T_s}) - 3(z-1)(z-e^{-3T_s}) + 2(z-1)(z-e^{-2T_s})}{(z-1)(z-e^{-2T_s})(z-e^{-3T_s})}\right)$$

Expanding the numerator, the z^2 terms cancel ($z^2 - 3z^2 + 2z^2 = 0$), which is something you should always expect to see here. Using $e^n e^m = e^{n+m}$ and collecting the remaining terms gives:

$$= \frac{2z}{z-1}\left(\frac{(1 - 3e^{-2T_s} + 2e^{-3T_s})z + (2e^{-2T_s} - 3e^{-3T_s} + e^{-5T_s})}{(z-e^{-2T_s})(z-e^{-3T_s})}\right)$$

With $T_s = 1$ s, $e^{-2} = 0.1353$, $e^{-3} = 0.0498$ and $e^{-5} = 0.0067$, this gives:

$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{2z}{z-1}\left(\frac{0.6936z + 0.1280}{(z-0.1353)(z-0.0498)}\right)$$

Step 4: Multiply by the ZOH numerator

$$G(z) = \frac{z-1}{z} \cdot \frac{2z}{z-1}\left(\frac{0.6936z + 0.1280}{(z-0.1353)(z-0.0498)}\right) = \frac{2(0.6936z + 0.1280)}{(z-0.1353)(z-0.0498)}$$

Expanding gives:

$$G(z) = \frac{1.3871z + 0.2561}{z^2 - 0.1851z + 0.0067}$$

Note that the continuous-time poles at $s = -2$ and $s = -3$ have become discrete-time poles at $z = e^{-2T_s} = 0.1353$ and $z = e^{-3T_s} = 0.0498$, as expected from $z = e^{sT_s}$.

Using MATLAB to Check Example 1

MATLAB Script: discretising the transfer function in Example 1

```
clear; clc; close all;

%% Continuous-time transfer function
G = tf(12, [1 5 6]);

%% Discrete-time transfer function
T_s = 1;          % sampling interval [s]
Gd = c2d(G, T_s)  % ZOH method by default

%% Back to continuous time
Gc = d2c(Gd)      % ZOH method by default
```

After clicking **Run**, the following appears in the Command Window, which agrees with Example 1. Converting back with `d2c` recovers the original $G(s)$.

```
Gd =
      1.387 z + 0.2561
  ------------------------
  z^2 - 0.1851 z + 0.006738

Sample time: 1 seconds
Discrete-time transfer function.

Gc =
       12
  -------------
  s^2 + 5 s + 6

Continuous-time transfer function.
```

6.0 Exercises

Exercise 1:

i. Sketch and label a closed-loop sampled-data control system with negative unity feedback, and relate it to a practical digital control system.

ii. Describe the main differences between a continuous-time control system and a sampled-data control system.

iii. Starting from the ZOH transfer function $G_{ZOH}(s) = (1 - e^{-sT_s})/s$, show that the discrete-time equivalent of a continuous-time system $G(s)$ is $G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$.

iv. Explain the operation of the ZOH in a sampled-data control system.

Fully worked solutions are given in Appendix A.

Exercise 2: for the following continuous-time transfer functions, determine the discrete-time transfer function, $G(z)$, with a ZOH and a sampling interval $T_s = 1$ s:

a) $G(s) = \frac{2}{(s+1)(s+3)}$

b) $G(s) = \frac{2}{s^2+6s+8}$

c) $G(s) = \frac{10}{s^2+2s+1}$

Check your answers using the MATLAB script in Section 5.0. Answers are given in Appendix B.

7.0 Summary

This Learning Guide introduced the sampled-data control system, in which a digital control algorithm controls a continuous-time process. The sampling switch and ADC convert continuous signals into samples, and the DAC and zero-order hold (ZOH) convert the controller's output back into a continuous staircase signal. The sampling interval, T_s, is a compromise between performance, computational load and the risk of aliasing.

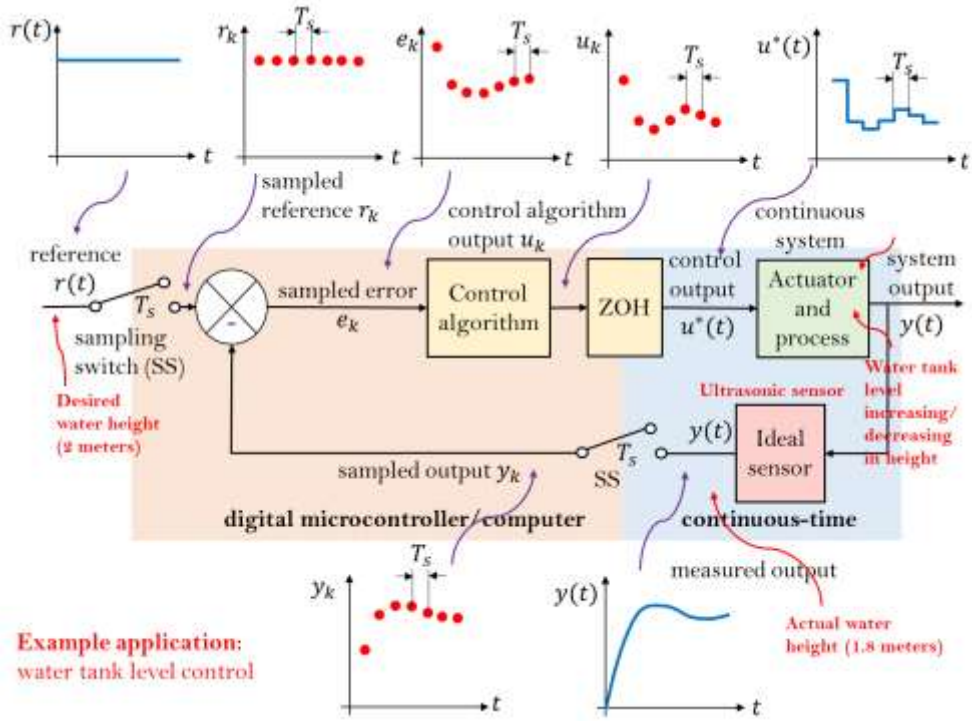
The z-transform was introduced as the discrete-time equivalent of the Laplace transform, with z^{-1} representing a delay of one sampling interval and $z = e^{sT_s}$ linking the two domains. The transfer function of the ZOH was derived, and used to obtain the key result $G(z) = \frac{z-1}{z}\mathcal{Z}\{G(s)/s\}$. A continuous-time transfer function was then discretised in four steps (partial fractions, table of transforms, common denominator and multiplying by the ZOH numerator), and the result was checked using MATLAB. The key progression introduced in this guide can be summarised as:

$G(s) \rightarrow G(s)/s$ in partial fractions $\rightarrow$ table of transforms $\rightarrow$ multiply by $(z-1)/z \rightarrow G(z)$

Appendix A: Solutions to Exercise 1

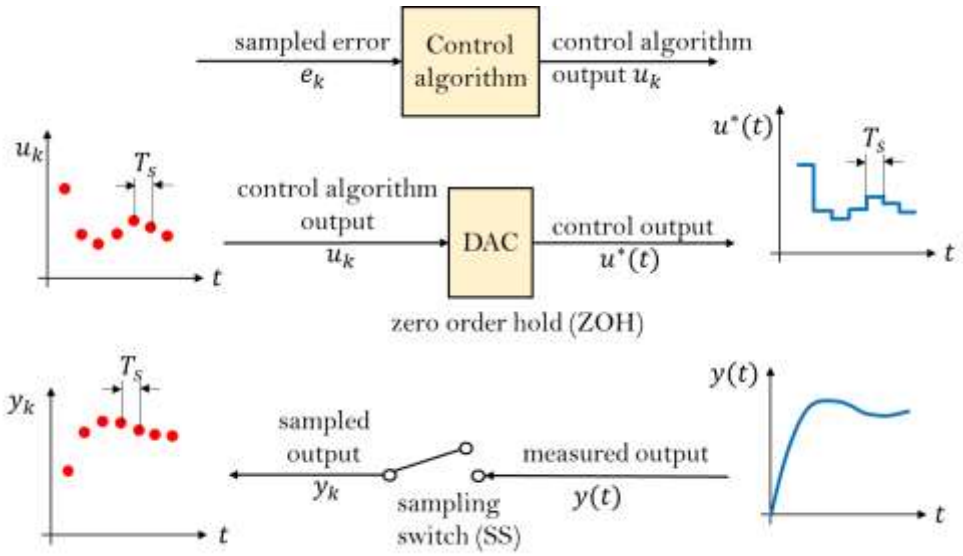
i. Sampled-data control system

An example is water tank level control. The control algorithm runs on a microcontroller, which samples the measured level from an ultrasonic sensor, and the ZOH drives the valve between samples:



ii. Continuous-time and sampled-data control systems

A continuous-time controller is a hardwired electronic circuit (for example, an arrangement of operational amplifiers implementing a PID controller), which acts on the signals continuously. In a sampled-data control system, the controller is a control algorithm running on a microcontroller or computer. It only acts at the sampling instants, so it requires a sampling switch (SS) and ADC to obtain samples of the measurement, and a DAC and zero-order hold (ZOH) to produce a continuous control signal:



As a result, the behaviour of the closed-loop control system depends on the sampling interval, T_s, as well as on the controller gains.

iii. Discrete-time equivalent of $G(s)$

From $f(t) = e^{-at}$, the s-domain pole at $s = -a$ becomes the z-domain pole at $z = e^{-aT_s}$, so $z = e^{sT_s}$ and therefore $z^{-1} = e^{-sT_s}$. Replacing e^{-sT_s} in the numerator of the ZOH gives:

$$1 - e^{-sT_s} = 1 - z^{-1} = \frac{z-1}{z}$$

which is the z-form of the ZOH numerator. Therefore:

$$G(z) = \mathcal{Z}\{G_{ZOH}(s)\,G(s)\} = \mathcal{Z}\left\{\frac{1-e^{-sT_s}}{s}G(s)\right\} = \frac{z-1}{z}\,\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

iv. Operation of the ZOH

The ZOH holds each controller output, u_k, constant for one sampling interval, turning the sequence of samples into the staircase signal $u^*(t)$ that drives the actuator. Each held value is a pulse made of a step applied at the start of the interval and the same step removed one sampling interval later, giving the transfer function:

$$G_{ZOH}(s) = \frac{1}{s} - \frac{1}{s}e^{-sT_s} = \frac{1 - e^{-sT_s}}{s}$$

where e^{-sT_s} is a delay of one sampling interval. The hold introduces an average delay of about half a sampling interval, which can reduce stability if T_s is too long.

Appendix B: Answers to Exercise 2

Table 3: *Answers to Exercise 2 (ZOH, T_s = 1 s)*

Question	$G(s)$	$G(z)$
a	$\frac{2}{(s+1)(s+3)}$	$\frac{0.3154z + 0.0850}{z^2 - 0.4177z + 0.0183}$
b	$\frac{2}{s^2 + 6s + 8}$	$\frac{0.1869z + 0.0253}{z^2 - 0.1537z + 0.0025}$
c	$\frac{10}{s^2 + 2s + 1}$	$\frac{2.6424z + 1.3534}{z^2 - 0.7358z + 0.1353}$

For (c), $G(s) = 10/(s+1)^2$ has a repeated pole, so the partial fractions of $G(s)/s$ include a $1/(s+1)^2$ term, which is converted using Row 8 of Table 2.

Recommended Citation

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