On the utilization of Macroscopic Information for String Stability of a Vehicular Platoon

Marco Mirabilio, Alessio Iovine, Elena De Santis, Maria Domenica Di Benedetto, Giordano Pola

***Abstract*— The use of macroscopic information to improve control performance of a vehicular platoon composed of autonomous vehicles is investigated. A mesoscopic control law is provided, and a String Stability analysis is performed using Lyapunov functions and Input-to-State Stability (ISS) concepts. Simulations are implemented in order to validate the controller and to show the efficacy of the proposed approach for mitigating traffic oscillations.**

Keywords: String Stability, Input-to-State Stability, platoon control, mesoscopic modeling, Cooperative Adaptive Cruise Control.

I. Introduction

Nowadays, Vehicle-to-Infrastucture (V2I) and Vehicle-to-Vehicle (V2V) communication technologies are a reality in the smart transportation domain (see [1]), and their utilization in Cooperative Adaptive Cruise Control (CACC) is widely expected to improve traffic conditions (see [2], [3], [4]). Indeed, traffic jamming transition has been shown to strongly depend on the amplitude of fluctuations of the leading vehicle (see [5]), and interconnected autonomous vehicles are sensed to reduce stop-and-go waves propagation and traffic oscillations via the concept of String Stability [2], [6], [7], [8].

String Stability relies on the idea that disturbances acting on an agent of the cluster should not amplify backwards in the string. In the case of vehicular platooning, disturbances may be due to reference speed variation, external inputs acting on each vehicle, wrong modeling, etc. Several cases of information sharing have been considered for each leader-follower interaction, but a common characteristic is that some microscopic variables are always shared among the whole platoon, e.g. the acceleration of the platoon's leading vehicle (see [6]) or its desired speed profile (see [3]). It needs a V2V communication among the whole platoon, or a V2I bidirectional exchange of information. This paper analyses the benefits of the information propagation in a String Stability framework using both microscopic and macroscopic information for control purposes. Each follower is here considered to correctly measure the distance and speed of its leading vehicles, using for example radar and LIDAR. To improve control performance, macroscopic information is supposed to be obtained and communicated either from the road infrastructure (V2I) or from the whole platoon (V2V). The leader acceleration is communicated only to its predecessor. We target a platoon composed by autonomous vehicles implementing CACC, but the framework is suitable for including autonomous vehicles implementing simple ACC or even human-driven vehicles as part of the platoon.

The framework we propose is based on sharing macroscopic quantities along the platoon. The use of those quantities aims at increasing the ability of each car-following situation to counteract the disturbances by providing an anticipatory behaviour capable to absorb traffic jam. The idea of using macroscopic quantities, mainly the density, for microscopic traffic control has already been introduced in the literature, resulting in a mesoscopic modeling. In [9], [10] and in the references therein, the focus is on simulation aspects and real data analysis. Several works are now focusing on a mesoscopic modeling for traffic control purposes (see [11], [12] [13], [14]).

The controller we propose considers macroscopic information and ensures Asymptotic String Stability. The adopted nonlinear spacing policy relies on the family of nonlinear spacing strategies introduced in [15] and [16]. Similarly to [3], the result is obtained through an inductive method exploiting Input-to-State Stability (ISS). The main difference is that ISS is ensured with respect to the leader-follower situation and the ahead vehicles of each leader. Simulations show the improvements on the whole traffic throughput producing an anticipatory behaviour and oscillations reduction, and providing a better transient harmonization while maintaining String Stability properties.

The paper is organized as follows. Section II introduces the considered framework, while Section III the needed control tools. Control laws are derived and stability analysis is performed in Section IV. Simulations are carried out in Section V. Some concluding remarks are outlined in Section VI.

Notation - $\mathbb{R}^+$ is the set of non-negative real numbers. For a vector $x \in \mathbb{R}^n$, $|x| = \sqrt{x^T x}$ is its Euclidean norm. The $\mathcal{L}_\infty$ signal norm is defined as $|x(\cdot)|_\infty^{[t_0,t]} = \sup_{t_0 \leq \tau \leq t} |x(\tau)|$. The $p = \infty$ norm of a vector is denoted by $|x|_\infty = \max_{i=1\dots n} |x_i|$. If a different norm is used, it is indicated by a subscript (e.g. $|x|_p$ denotes the generic p norm). We refer to [17] for the definition of Lyapunov functions, and functions $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$.

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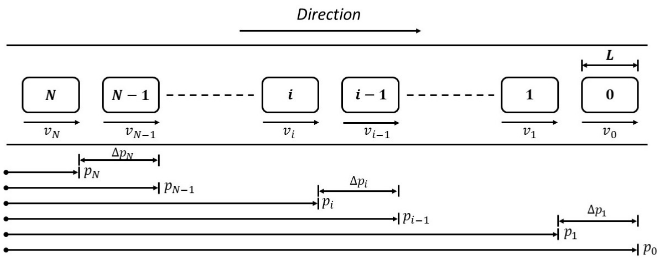

Fig. 1. Reference framework.

II. Modeling and String Stability definitions

A. Platoon modeling

We consider a cluster of $N+1$ vehicles, $N \in \mathbb{N}$, proceeding in the same direction on a single lane road, as in Fig.1. We make the following

Assumption 1: All the vehicles are equal, with the same length $L \in \mathbb{R}^+$ and have the common goal of collision avoidance, i.e. to maintain a strictly positive distance among them, while keeping the same speed.
We denote with $i = 0$ the first vehicle of the platoon and with $\mathcal{I}_N = \{1, 2, ..., N\}$ the set of follower vehicles. The set including all the vehicles is $\mathcal{I}_N^0 = \mathcal{I}_N \cup \{0\}$.

Similarly to [3], each vehicle $i \in \mathcal{I}_N^0$ is assumed to satisfy the following longitudinal dynamics:

$$\begin{aligned} \dot{p}_i &= f_p(\xi_i) \\ \dot{\xi}_i &= f_\xi(\xi_i) + g_\xi(\xi_i) u_i \end{aligned} \tag{1}$$

where $p_i \in \mathbb{R}^+$ is the position of vehicle i, $v_i = \dot{p}_i$ ($0 < v_i \le v_{\max}$, $v_{\max} \in \mathbb{R}^+$) is its velocity and the acceleration u_i is the vehicle control input. Variable $\xi_i \in \mathbb{R}^{n-1}$ represents the remaining dynamics of the vehicle, such as actuators dynamics. Since the reaction delay has only slight quantitative influence on the oscillation growth pattern (see [18]), no delays are considered here either for the reaction time or the communication time.

According to [6] and [19], the dynamics in (1) can be simplified:

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= u_i \end{aligned} \tag{2}$$

where $\xi_i = v_i$, the functions $f_p(\xi_i) = v_i$, $f_\xi(\xi_i) = 0$ and $g_\xi(\xi_i) = 1$; u_i is the acceleration of vehicle i ($|u_i| \le a_{\max}, a_{\max} \in \mathbb{R}^+$). To provide a global description of the platoon, we adopt the leader-follower model that describes the inter-vehicular interaction (see [20]). We define the state of each vehicle $i \in \mathcal{I}_N^0$ as

$$x_i = [p_i \quad v_i]^T \tag{3}$$

and the state of each car-following situation among the leading vehicle $i-1$ and the following one i as

$$\chi_i = x_i - x_{i-1} = \left[\begin{array}{c} \Delta p_i \\ \Delta v_i \end{array} \right] = \left[\begin{array}{c} p_i - p_{i-1} \\ v_i - v_{i-1} \end{array} \right]. \tag{4}$$

Positions, speed and acceleration of each leading vehicle are supposed to be perfectly known, either measured or communicated to the following one. Consequently, the obtained dynamical model is:

$$\dot{\chi}_i = \left[\begin{array}{c} \Delta \dot{p}_i \\ \Delta \dot{v}_i \end{array} \right] = \left[\begin{array}{c} \Delta v_i \\ u_i - u_{i-1} \end{array} \right], \quad i \in \mathcal{I}_N, \tag{5}$$

or, shortly,

$$\dot{\chi}_i = f(\chi_i, u_i, u_{i-1}), \quad i \in \mathcal{I}_N. \tag{6}$$

The model can be extended with the inclusion of a disturbance term, representing modeling errors, the missing modeling of the delays in (1) or external disturbances acting on the vehicle (see [3]).

To derive the dynamics of the first vehicle $i = 0$ of the platoon, in the same form of (5), a non-autonomous non-communicating virtual leader $i = -1$ is considered to precede the set of vehicles, with dynamical model

$$\dot{x}_{-1} = \left[\begin{array}{c} \dot{p}_{-1} \\ \dot{v}_{-1} \end{array} \right] = \left[\begin{array}{c} v_{-1} \\ u_{-1} \end{array} \right]. \tag{7}$$

Then, the car-following dynamics with respect to vehicle $i = 0$ can be described by:

$$\dot{\chi}_0 = \left[\begin{array}{c} \Delta \dot{p}_0 \\ \Delta \dot{v}_0 \end{array} \right] = \left[\begin{array}{c} \Delta v_0 \\ u_0 - u_{-1} \end{array} \right]. \tag{8}$$

It follows that dynamics in (6) is valid $\forall\ i \in \mathcal{I}_N^0$. Since we consider $i = -1$ to represent a virtual vehicle, $p_{-1}(t) = \int_0^t v_{-1}(\tau) d\tau,\ t \ge 0$, is a dummy state. Moreover, we consider $\Delta p_0(t) = -\Delta \bar{p},\ \forall t \ge 0$, where $\Delta \bar{p} > 0$ is the constant desired inter-vehicular distance. Since $p_i < p_{i-1}$, the desired distance has to be negative.

In accordance to [20], [21], [22], we consider the widely accepted hypothesis of constant speed for the virtual leader $i = -1$, that precedes the entire cluster. Then, we have

$$p_{-1}(t) = \bar{v} \cdot t,\ v_{-1}(t) = \bar{v},\ u_{-1}(t) = 0,\ \forall t \ge 0 \tag{9}$$

where $\bar{v} > 0$ is a constant speed. For vehicular platoons, the constant speed assumption defines the equilibrium point for all the vehicles in the cluster. Consequently, when all $i \in \mathcal{I}_N^0$ have equal speed and are at the same desired distance $\Delta \bar{p}$, the equilibrium point for the i-th system of dynamics (6) is

$$\chi_{e,i} = \bar{\chi} = [\ -\Delta \bar{p} \quad 0\ ]^T. \tag{10}$$

Let χ be the lumped state of the entire vehicle platoon:

$$\chi = [\chi_0^T\ \chi_1^T\ ...\ \chi_N^T]^T. \tag{11}$$

Then, for $u_0 = 0$ it follows that

$$\chi_e = [\bar{\chi}^T\ \bar{\chi}^T\ ...\ \bar{\chi}^T]^T. \tag{12}$$

B. String Stability definitions

Let the model in (5) describe a platoon, and its equilibrium be (10). The input u_i is generated by the following dynamic controller

$$\begin{cases} \dot{\vartheta}_i = \omega_i(\vartheta_i, \chi, \chi_{e,i}) \\ u_i = h_i(\vartheta_i, \chi, \chi_{e,i}, u_{i-1}) \end{cases} \tag{13}$$

taking as inputs $\chi,\ \chi_{e,i},\ u_{i-1}$. The resulting closed loop system is denoted in the sequel by P_{cl}. We recall the notions of String Stability and Asymptotic String Stability according to [6] and [3].

Definition 1: (String Stability) The equilibrium $\chi_{e,i} = \bar{\chi},\ i \in \mathcal{I}_N^0$, of P_{cl} is said to be String Stable if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $N \in \mathbb{N}$,

$$\max_{i\in\mathcal{I}_N^0} |\chi_i(0)-\bar{\chi}| < \delta \Rightarrow \max_{i\in\mathcal{I}_N^0} |\chi_i(t)-\bar{\chi}| < \epsilon,\ \forall\, t \geq 0. \quad (14)$$

Definition 2: (Asymptotic String Stability) The equilibrium $\chi_{e,i} = \bar{\chi},\ i \in \mathcal{I}_N^0$, of P_{cl} is said to be Asymptotically String Stable if it is String Stable and, for all $N \in \mathbb{N}$,

$$\lim_{t\to\infty} |\chi_i(t) - \bar{\chi}| = 0, \quad \forall\ i \in \mathcal{I}_N^0. \quad (15)$$

III. Control tools

The goal of this paper is to design a control law as in (13) that adopts mesoscopic quantities for ensuring asymptotic string stability of P_{cl}. To this purpose, a proper spacing policy and a function describing macroscopic information are introduced.

A. Spacing policy

Several spacing policies have been introduced in the literature (see [23], [24]). We adopt a variable time spacing policy, which consists in tracking a variable inter-vehicular desired distance and allows for string stability and a low inter-vehicular spacing at steady-state (see [15] and [16]). We define a *mesoscopic* time varying trajectory for the distance policy Δp_i^r of the i-th vehicle with respect to its leader $i-1$:

$$\Delta p_i^r(t) = -\Delta\bar{p} - \rho_{i-1}^M(t),\ t \geq 0 \quad (16)$$

where $\Delta\bar{p} > 0$ is the desired constant inter-vehicular distance and $\rho_{i-1}^M(t)$ is a function describing macroscopic information. Our goal is to show that, by using the macroscopic information, transient harmonization when traffic conditions vary is obtained while maintaining the platoon equilibrium in (12) in steady-state.

B. Macroscopic information

Here we define proper macroscopic functions taking into account microscopic distance and speed variance, similarly to [9]. Given the generic vehicle $i \in \mathcal{I}_N$, let $\mu_{\Delta p,i}$ and $\sigma^2_{\Delta p,i}$ be the inter-vehicular distance mean and variance computed from vehicle 1 to vehicle i, respectively:

$$\mu_{\Delta p,i} = \frac{1}{i+1}\sum_{j=0}^{i} \Delta p_j, \quad \sigma^2_{\Delta p,i} = \frac{1}{i+1}\sum_{j=0}^{i}(\Delta p_j - \mu_{\Delta p_i})^2. \quad (17)$$

Let $\mu_{\Delta v,i}$ and $\sigma^2_{\Delta v,i}$ be the velocity tracking error mean and variance computed from vehicle 1 to vehicle i, respectively:

$$\mu_{\Delta v,i} = \frac{1}{i+1}\sum_{j=0}^{i} \Delta v_j, \quad \sigma^2_{\Delta v,i} = \frac{1}{i+1}\sum_{j=0}^{i}(\Delta v_j - \mu_{\Delta v_i})^2. \quad (18)$$

We denote by $\psi^i_{\Delta p} : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ the distance macroscopic function and by $\psi^i_{\Delta v} : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ the speed tracking error macroscopic function. Then, we propose the macroscopic functions

$$\psi^i_{\Delta p} = \gamma_{\Delta p} sign(\Delta\bar{p} + \mu_{\Delta p,i})\sqrt{\sigma^2_{\Delta p,i}}, \quad (19)$$

$$\psi^i_{\Delta v} = \gamma_{\Delta v} sign(\mu_{\Delta v,i})\sqrt{\sigma^2_{\Delta v,i}}, \quad (20)$$

where $\gamma_{\Delta p}, \gamma_{\Delta v} > 0$ are constant parameters, and $\mu_{\Delta p,i}$, $\mu_{\Delta v,i}$, $\sigma^2_{\Delta p,i}$ and $\sigma^2_{\Delta v,i}$ are defined in (17) and (18). The sign function is defined as

$$sign(y) = \begin{cases} 1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0 \end{cases} \quad (21)$$

Functions (19) and (20) connect the macroscopic density function with the variance of the microscopic distance and speed difference. Instead of considering the whole set of leader-follower situations, they allow for a complexity reduction of the considered interconnection framework without reducing the level of available information.
We embed the macroscopic information given by (19) and (20) in the macroscopic function denoted

$$\rho_i = [\rho_{1,i} \quad \rho_{2,i}]^T, \quad (22)$$

described by the asymptotically stable dynamical system:

$$\begin{cases} \dot{\rho}_{1,i} = -\lambda_1\rho_{1,i} + \rho_{2,i} \\ \dot{\rho}_{2,i} = -\lambda_2\rho_{2,i} + (a\psi^i_{\Delta p} + b\psi^i_{\Delta v}) \\ \rho_{1,i}(0) = \rho_{2,i}(0) = 0 \end{cases} \quad (23)$$

where $a, b \geq 0$ are chosen parameters, and $\lambda_1, \lambda_2 > 0$ represents the forgetting factors allowing $\rho_i(t)$ to have memory only of the recent evolution of the system. Different macroscopic functions can be proposed, as in [14]. The main characteristic of ρ_i in (23) is to incorporate the whole macroscopic information of the platoon avoiding complexity calculation explosion by the control law due to the state explosion.

If the functions (17) and (18) were to be obtained through V2V communications, a demanding exchange of information could be necessary. However, those functions can be easily calculated by the road infrastructure, and then communicated to the vehicles via V2I communications. Consequently, the function ρ_i in (23) can be obtained without V2V communication.

IV. Mesoscopic control law

In this section, the control law adopting mesoscopic quantities for a single car-following situation is introduced. Then, String Stability and Asymptotic String Stability w.r.t. Definitions 1 and 2 is ensured when the control laws are implemented for each leader-follower situation along the platoon. The control law implements the variable spacing policy in (16) while considering the function ρ_i in (23). Each vehicle is modeled according to dynamics (2) and each car-following situation according to χ_i in (5) and (8). To analyze the String Stability of the closed loop system, we consider an extended state of (4) that includes the dynamics in (23):

$$\hat{\chi}_i = \left[\ \Delta p_i \quad \Delta v_i \quad \rho_{i-1}^T\ \right]^T,\ \forall\ i \in \mathcal{I}_N^0. \quad (24)$$

Defining $\rho_{-1} = g_{\rho,-1}(\hat{\chi}_{-1}) = 0$, we can describe the closed loop dynamics of (24) as

$$\dot{\hat{\chi}}_0 = f_{cl}(\hat{\chi}_0),\ i = 0, \tag{25}$$

$$\dot{\hat{\chi}}_i = f_{cl}(\hat{\chi}_i) + g_{cl,i}(\hat{\chi}_{i-1}, \hat{\chi}_{i-2}, ..., \hat{\chi}_0),\ \forall\ i \in \mathcal{I}_N. \tag{26}$$

where $f_{cl} : \mathbb{R}^4 \to \mathbb{R}^4$ is the vector field describing the evolution dynamics of each isolated subsystem, and $g_{cl,i} : \underbrace{\mathbb{R}^4 \times \cdots \times \mathbb{R}^4}_{i \text{ times}} \to \mathbb{R}^4$ is the interconnection term. We assume that the virtual leader $i = -1$ has a constant speed $v_{-1} = \bar{v} > 0,\ u_{-1} = 0$. The assumption of $\Delta p_0(t) = -\Delta\bar{p},\ \forall t \geq 0$ is considered.

A. Control strategy for variable spacing policy

We present a control law for implementing the variable spacing policy in (16), with $\rho^M = \rho_{1,i}$. The controller associated to the $i-$th vehicle, $\forall\ i \in \mathcal{I}_N^0$, is given by:

$$\begin{aligned} u_i &= u_{i-1} - (\Delta p_i + \Delta\bar{p} + \rho_{1,i-1}) - \lambda_1(\lambda_1\rho_{1,i-1} - \rho_{2,i-1}) \\ &\quad + \lambda_2\rho_{2,i-1} - K_{\Delta p}(\Delta v_i - \lambda_1\rho_{1,i-1} + \rho_{2,i-1}) \\ &\quad - a\psi_{\Delta p}^{i-1} - b\psi_{\Delta v}^{i-1} - K_{\Delta v}(\Delta v_i - \lambda_1\rho_{1,i-1} + \rho_{2,i-1} \\ &\quad + K_{\Delta p}(\Delta p_i + \Delta\bar{p} + \rho_{1,i-1})) \\ &= u_{i-1} - (\Delta p_i - \Delta p_i^r) - K_{\Delta v}(\Delta v_i - \Delta v_i^r) \\ &\quad + (K_{\Delta p} - \lambda_1)(\lambda_1\rho_{1,i-1} - \rho_{2,i-1}) + \lambda_2\rho_{2,i-1} \\ &\quad - K_{\Delta p}\Delta v_i - a\psi_{\Delta p}^{i-1} - b\psi_{\Delta v}^{i-1}, \end{aligned} \tag{27}$$

with

$$\Delta v_i^r = \lambda_1\rho_{1,i-1} - \rho_{2,i-1} - K_{\Delta p}(\Delta p_i - \Delta p_i^r) \tag{28}$$

and given constant gains $K_{\Delta p}, K_{\Delta v} > 0$ equal for each $i \in \mathcal{I}_N^0$, $\Delta p_i^r = -\Delta\bar{p} - \rho_{1,i-1}$, and $\rho_{1,i}, \rho_{2,i}$ as defined in (23).

We remark that the controller in (27) considers both microscopic and macroscopic information, resulting to describe a *mesoscopic* framework. To analyze the String Stability of the closed loop system, we consider the extended leader-follower state vector $\hat{\chi}_i$ in (24). Its equilibrium point is

$$\hat{\chi}_{e,i} = [\ -\Delta\bar{p}\ \ 0\ \ 0\ \ 0\ ]^T,\ \forall i \in \mathcal{I}_N^0. \tag{29}$$

The closed loop dynamics with respect to (8) and (5) results to be, respectively:

$$\dot{\hat{\chi}}_0 = \begin{bmatrix} \Delta\dot{p}_0 \\ \Delta\dot{v}_0 \\ \dot{\rho}_{1,-1} \\ \dot{\rho}_{2,-1} \end{bmatrix} = \begin{bmatrix} \Delta v_0 \\ -(K_{\Delta p} + K_{\Delta v})\Delta v_0 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

$$\dot{\hat{\chi}}_i = \begin{bmatrix} \Delta v_i \\ (*) \\ -\lambda_1\rho_{1,i-1} + \rho_{2,i-1} \\ -\lambda_2\rho_{2,i-1} + a\psi_{\Delta p}^{i-1} + b\psi_{\Delta v}^{i-1} \end{bmatrix} \tag{31}$$

with

$$\begin{aligned} (*) &= -(\Delta p_i - \Delta p_i^r) - K_{\Delta v}(\Delta v_i - \Delta v_i^r) \\ &\quad + (K_{\Delta p} - \lambda_1)(\lambda_1\rho_{1,i-1} - \rho_{2,i-1}) + \lambda_2\rho_{2,i-1} \\ &\quad - K_{\Delta p}\Delta v_i - a\psi_{\Delta p}^{i-1} - b\psi_{\Delta v}^{i-1}, \end{aligned}$$

We can rewrite the system in (30) and (31) as (25) and (26), where $g_{cl,i}(\hat{\chi}_{i-1}, ..., \hat{\chi}_0)$ is

$$g_{cl,i}(\hat{\chi}_{i-1}, \hat{\chi}_{i-2}, ..., \hat{\chi}_0) = \begin{bmatrix} 0 \\ -(a\psi_{\Delta p}^{i-1} + b\psi_{\Delta v}^{i-1}) \\ 0 \\ a\psi_{\Delta p}^{i-1} + b\psi_{\Delta v}^{i-1} \end{bmatrix}. \tag{32}$$

B. String Stability analysis

Set $\tilde{\chi}_i = \hat{\chi}_i - \hat{\chi}_{e,i}$. Then, the following result holds:

Lemma 1: Consider the closed loop system described by (30) and (31). Then, there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}_\infty$ such that, if $K_{\Delta p}, K_{\Delta v}, \lambda_1, \lambda_2 > 0$ then,

$$|\tilde{\chi}_i(t)| \leq \beta(|\tilde{\chi}_i(0)|, t) + \gamma\left(\max_{j=0,...,i-1} |\tilde{\chi}_j(\cdot)|_\infty^{[0,t]}\right) \tag{33}$$

$\forall t \geq 0$, and $\gamma(s) = \tilde{\gamma}s,\ \ s \geq 0,\ \tilde{\gamma} \in \mathbb{R}^+$. Moreover, there exist a and b in (23) such that $\tilde{\gamma} \in (0, 1)$.

Proof: Let us consider the candidate Lyapunov function (see [17]) $W_i = W(\tilde{\chi}_i)$ for the i-th dynamical system $\tilde{\chi}_i$, for $i \in \mathcal{I}_N^0$:

$$\begin{aligned} W(\tilde{\chi}_i) &= \frac{1}{2}(\Delta p_i - \Delta p_i^r)^2 + \frac{1}{2}(\Delta v_i - \Delta v_i^r)^2 \\ &\quad + \frac{1}{2}\rho_{1,i-1}^2 + \frac{1}{2}\rho_{2,i-1}^2. \end{aligned} \tag{34}$$

It satisfies

$$\underline{\alpha}|\tilde{\chi}_i|^2 \leq W(\tilde{\chi}_i) \leq \bar{\alpha}|\tilde{\chi}_i|^2, \tag{35}$$

where

$$\underline{\alpha} = \frac{1}{2}, \quad \bar{\alpha} = \frac{1}{2}\max\{1 + K_{\Delta p}^2, 2 + (\lambda_1 - K_{\Delta p})^2\}, \tag{36}$$

(see Appendix A for more details). Then, the time derivative of W_i in (34) verifies:

$$\begin{aligned} \dot{W}_i &= -K_{\Delta p}(\Delta p_i - \Delta p_i^r)^2 - K_{\Delta v}(\Delta v_i - \Delta v_i^r)^2 \\ &\quad - \lambda_1\rho_{1,i-1}^2 - \lambda_2\rho_{2,i-1}^2 + \rho_{1,i-1}\rho_{2,i-1} \\ &\quad + \rho_{2,i-1}(a\psi_{\Delta p}^{i-1} + b\psi_{\Delta v}^{i-1}) \\ &\leq -\alpha|\tilde{\chi}_i|^2 + |\tilde{\chi}_i|(a|\psi_{\Delta p}^{i-1}| + b|\psi_{\Delta v}^{i-1}|) \end{aligned} \tag{37}$$

where

$$\alpha = \min\{c_1, c_2, c_3, c_4\}, \tag{38}$$

$c_1 = K_{\Delta p}(1 + K_{\Delta p}K_{\Delta v}),\ c_2 = K_{\Delta v},\ c_3 = K_{\Delta p} + \lambda_1 + K_{\Delta v}(\lambda_1 - K_{\Delta p})^2,\ c_4 = \lambda_2 + K_{\Delta v}$. See Appendix B for more details on the calculations in (37). Let $l \in \{1, ..., m\}$ and $y_l \in \mathbb{R}$. Then, the variance with respect to the set of values y_l satisfies the property

$$\sigma_y^2 \leq \frac{1}{4}(\max_l y_l - \min_l y_l)^2. \tag{39}$$

From (39) it follows that

$$a|\psi_{\Delta p}^i| + b|\psi_{\Delta v}^i| \leq (a\gamma_{\Delta p} + b\gamma_{\Delta v}) \max_{j=0,...,i} |\tilde{\chi}_j|. \tag{40}$$

See Appendix C for more details on the calculations in (40). Define

$$d = a\gamma_{\Delta p} + b\gamma_{\Delta v} > 0, \quad \Upsilon \in (0, 1). \tag{41}$$

Then

$$\dot{W}_i \le -\alpha|\tilde{\chi}_i|^2 + d|\tilde{\chi}_i| \max_{j=0,\dots,i-1} |\tilde{\chi}_j| + \Upsilon\alpha|\tilde{\chi}_i|^2 - \Upsilon\alpha|\tilde{\chi}_i|^2$$
$$\le -(1-\Upsilon)\alpha|\tilde{\chi}_i|^2, \quad \forall\, |\tilde{\chi}_i| \ge \frac{d}{\alpha\Upsilon} \max_{j=0,\dots,i-1} |\tilde{\chi}_j|. \tag{42}$$

Since $\alpha > 0$, the inequality in (42) satisfies the Input-to-State Stability (ISS) condition (see [17]). According to [17, Theorem 4.19], the inequality in (33) is verified. Moreover,

$$\gamma(s) = \tilde{\gamma} s \ \forall\ s \ge 0, \ \tilde{\gamma} = \sqrt{\frac{\bar{\alpha}}{\underline{\alpha}}} \frac{d}{\alpha\Upsilon} > 0. \tag{43}$$

Since the parameters $a, b \ge 0$ in the dynamics of ρ_i in (23) can be arbitrarily selected, the constant d defined in (41) can be chosen such that $\tilde{\gamma}$ in (43) belongs to $(0, 1)$.

■

On the basis of Lemma 1, Asymptotic String Stability of the platoon can be obtained by using an appropriately chosen function describing macroscopic information, as shown in the following:

Theorem 1: The closed loop system described by (30) and (31) where the parameters $K_{\Delta p}, K_{\Delta v}, \lambda_1, \lambda_2 > 0$ and parameters a, b are such that $\tilde{\gamma} \in (0, 1)$, is Asymptotically String Stable.

Proof: The first part of the proof is based on the forward recursive application of the ISS property in Lemma (1) through an inductive method. For $i = 0$:

$$|\tilde{\chi}_0(t)| \le \beta(|\tilde{\chi}_0(0)|, t), \ \forall\ t \ge 0. \tag{44}$$

For $i = 1$:

$$|\tilde{\chi}_1(t)| \le \beta(|\tilde{\chi}_1(0)|, t) + \tilde{\gamma}|\tilde{\chi}_0(\cdot)|_\infty^{[0,t]}, \ \forall\ t \ge 0, \tag{45}$$

where $|\tilde{\chi}_0(\cdot)|_\infty^{[0,t]} \le \beta(|\tilde{\chi}_0(0)|, 0)$. Defining $|\tilde{\chi}_M(0)| = \max\{|\tilde{\chi}_0(0)|, |\tilde{\chi}_1(0)|\}$, then for both $i = 0$ and $i = 1$:

$$|\tilde{\chi}_0(t)| \le \beta(|\tilde{\chi}_M(0)|, 0), \ \forall\ t \ge 0, \tag{46}$$
$$|\tilde{\chi}_1(t)| \le \beta(|\tilde{\chi}_M(0)|, 0)(1 + \tilde{\gamma}), \ \forall t \ \ge 0. \tag{47}$$

For $i = 2$:

$$|\tilde{\chi}_2(t)| \le \beta(|\tilde{\chi}_2(0)|, t) + \tilde{\gamma} \max_{j=0,1} |\tilde{\chi}_j(\cdot)|_\infty^{[0,t]}, \ \forall\ t \ge 0. \tag{48}$$

Defining $|\tilde{\chi}'_M(0)| = \max_{j=0,1,2}\{|\tilde{\chi}_j(0)|\}$, since $\tilde{\gamma} > 0$, then

$$|\tilde{\chi}_0(t)| \le \beta(|\tilde{\chi}'_M(0)|, 0)(1 + \tilde{\gamma}), \ \forall\ t \ge 0, \tag{49}$$
$$|\tilde{\chi}_1(t)| \le \beta(|\tilde{\chi}'_M(0)|, 0)(1 + \tilde{\gamma}), \ \forall\ t \ge 0, \tag{50}$$

and

$$|\tilde{\chi}_2(t)| \le \beta(|\tilde{\chi}'_M(0)|, 0)(1 + \tilde{\gamma} + \tilde{\gamma}^2), \ \forall t \ge 0. \tag{51}$$

By recursively applying these steps, and since $\tilde{\gamma} \in (0, 1)$ for hypothesis, for each $i \in \mathcal{I}_N^0$ we state:

$$|\tilde{\chi}_i(t)| \le \beta\left(\max_{j=0,\dots,i} |\tilde{\chi}_j(0)|, 0\right) \sum_{j=0}^{i} \tilde{\gamma}^j$$
$$\le \beta\left(\max_{j=0,\dots,i} |\tilde{\chi}_j(0)|, 0\right) \sum_{j=0}^{\infty} \tilde{\gamma}^j$$
$$\le \frac{1}{1-\tilde{\gamma}} \beta\left(\max_{j=0,\dots,i} |\tilde{\chi}_j(0)|, 0\right), \ \forall\ t \ge 0. \tag{52}$$

Then

$$\max_{i \in \mathcal{I}_N^0} |\tilde{\chi}_i(t)| \le \frac{1}{1-\tilde{\gamma}} \beta\left(\max_{i \in \mathcal{I}_N^0} |\tilde{\chi}_i(0)|, 0\right), \ \forall\ t \ge 0. \tag{53}$$

Define $\omega(s) = \beta(s, 0), \ s \ge 0$. By definition of $\mathcal{KL}$ functions, ω is $\mathcal{K}_\infty$ and hence invertible. Since (53) holds for any $t \ge 0$, then

$$\delta = \omega^{-1}((1-\tilde{\gamma})\epsilon), \quad \forall\ \epsilon \ge 0. \tag{54}$$

The value of δ in (54) does not depend on the system dimension. From (52), (53) and (54), String Stability is ensured according to Definition 1.

We focus now on the possibility to ensure Asymptotic String Stability. This second part of the proof is based on a composition of Lyapunov functions (see [17]). We consider the function W_i associated with the i-th dynamical system, for $i \in \mathcal{I}_N^0$, that is described in (34) and satisfies the condition in (35).

Let us consider the time derivative of W_i in (37). By exploiting norm inequalities, it follows

$$a|\psi_{\Delta p}^{i-1}| + b|\psi_{\Delta v}^{i-1}| \le \sum_{j=0}^{i-1} \tilde{k}_j |\tilde{\chi}_j|. \tag{55}$$

See Appendix D for more details on the calculations in (55). Then

$$\dot{W}_i \le -\alpha|\tilde{\chi}_i|^2 + |\tilde{\chi}_i| \sum_{j=0}^{i-1} \tilde{k}_j |\tilde{\chi}_j|, \tag{56}$$

where

$$\tilde{k}_j = \sqrt{\frac{3}{i}} \max\{a\gamma_{\Delta p}, b\gamma_{\Delta v}\} > 0, \ j < i. \tag{57}$$

Let $\hat{\chi}$ and $\hat{\chi}_e$ be the extended lumped state of the platoon and the extended equilibrium point respectively, defined in a similar way as (11) and (12). Let us consider $\tilde{\chi} = \hat{\chi} - \hat{\chi}_e$ and the parameters $d_i > 0$ to define a composite function $W_c(\tilde{\chi})$:

$$W_c(\tilde{\chi}) = \sum_{i=0}^{N} d_i W(\tilde{\chi}_i). \tag{58}$$

It clearly verifies

$$\underline{\alpha}_c |\tilde{\chi}|^2 \le W_c(\tilde{\chi}) \le \bar{\alpha}_c |\tilde{\chi}|^2 \tag{59}$$

where

$$\underline{\alpha}_c = \min_{i \in \mathcal{I}_N^0} \{d_i\} \underline{\alpha}, \quad \bar{\alpha}_c = \max_{i \in \mathcal{I}_N^0} \{d_i\} \bar{\alpha}. \tag{60}$$

The time derivative of W_c in (58) satisfies the inequality

$$\dot{W}_c(\tilde{\chi}) \leq \sum_{i=0}^{N} d_i \left[-\alpha|\tilde{\chi}_i|^2 + \sum_{j=0}^{i-1} \tilde{k}_j |\tilde{\chi}_j||\tilde{\chi}_i| \right]. \quad (61)$$

Let us introduce the operator $\phi : \mathbb{R}^{2N+1} \to \mathbb{R}^{N+1}$, defined as

$$\phi(\tilde{\chi}) = [|\tilde{\chi}_0|, |\tilde{\chi}_1|, ..., |\tilde{\chi}_N|]^T. \quad (62)$$

Then, equation (61) can be rewritten as

$$\dot{W}_c(\tilde{\chi}) \leq -\frac{1}{2}\phi(\tilde{\chi})^T (DS + S^T D)\phi(\tilde{\chi}) \quad (63)$$

where

$$D = diag(d_0, d_1, ..., d_N) \quad (64)$$

and S is an $N \times N$ matrix whose elements are

$$s_{ij} = \begin{cases} \alpha & i = j \\ -\tilde{k}_j & i < j \\ 0 & i > j \end{cases} \quad (65)$$

For $\alpha > 0$, each leading principal minor of S is positive and hence it is an M-matrix. By [17, Lemma 9.7] there exists a matrix D such that $DS + S^T D$ is positive definite. Consequently, $\dot{W}_c$ in (63) is negative definite. It follows that W_c in (58) is a Lyapunov function for the overall platoon system described by (30) and (31). Therefor, there exists a $\mathcal{KL}$ function $\beta_c : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\tilde{\chi}(t)| \leq \beta_c(|\tilde{\chi}(0)|, t), \;\; \forall \; t \geq 0. \quad (66)$$

Condition in (66) ensures the asymptotic stability:

$$\lim_{t\to\infty} |\tilde{\chi}_i(t)| = 0, \quad \forall \; i \in \mathcal{I}_N^0. \quad (67)$$

The platoon system is proved to be String Stable by (53) and (54). Consequently, for each $i \in \mathcal{I}_N^0$ the state evolution $|\tilde{\chi}_i|$ is constrained by a bound that is independent from the system dimension. Furthermore, from (67) Asymptotic String Stability is ensured according to Definition 2.

■

V. Simulations

The introduced control strategy is simulated in Matlab&Simulink. Based on the modeling in (5), we consider a platoon of $N + 1 = 11$ vehicles. The initial conditions for each vehicle are randomly generated in a neighborhood of the equilibrium point. It results $\mu_{\Delta p} \neq \Delta\bar{p}, \mu_{\Delta v} \neq 0$ $\sigma^2_{\Delta p}, \sigma^2_{\Delta v} \neq 0$. The reference distance is $\Delta\bar{p} = 10m$ and the initial desired speed of the leading vehicle is $\bar{v} = 14m/s$. Vehicle speed $0 \leq v_i \leq 36$ $[m/s]$ and the acceleration is bounded such that $-4 \leq u_i \leq 4$ $[m/s^2]$. The control parameters are introduced in Table I, with a resulting $\tilde{\gamma} = 0.5$. To better stress the proposed controller, we analyze the behavior of the system when a disturbance acts on the acceleration of vehicle $i = 0$, and it is not communicated to vehicle $i = 1$. The simulation time is 1 minute. We split the simulation time in three phases:

TABLE I

THE CONTROL PARAMETERS.

Parameter	Value	Parameter	Value	Parameter	Value
$K_{\Delta p}$	1	$K_{\Delta v}$	2	Υ	0.9
λ_1, λ_2	1.5	a	0.6	b	0.6
$\gamma_{\Delta p}$	0.5	$\gamma_{\Delta v}$	0.5	$\tilde{\gamma}$	0.5

1) From $t = t_0 = 0s$ to $t = t_1 = 10s$: the vehicles start with initial conditions that are different from the desired speed and the desired distance. No disturbance is acting on the leader vehicle, and its desired speed is the initial one, i.e. $\bar{v} = 14m/s$.
2) From $t = t_1 = 10s$ to $t = t_2 = 30s$: a disturbance acts to the acceleration of the first vehicle $i = 0$. At t_1 a positive pulse of amplitude $4m/s^2$ and length $5s$ is considered, while a similar pulse with negative amplitude is considered at $t = 15s$. The control input of $i = 0$ being saturated, $i = 0$ succeeds to properly counteract to it but it is not able to operate the needed corrective action to return to the desired speed. Since the disturbance is an external input, it is not communicated to the follower and could propagate along the platoon.
3) From $t = t_2 = 30s$ to $t = t_3 = 60s$: the leader tracks a variable speed reference. From $t = 30s$ to $t = 45s$ the desired speed is $\bar{v} = 30m/s$, while from $t = 45s$ to $t = 60s$ it is $\bar{v} = 20m/s$.

Figures 2, 3 and 4 show, respectively, the inter-vehicular distance, speed and acceleration profiles for each vehicle the platoon when the control input in (27) is implemented. In this first phase, the vehicles are shown to quickly converge to the desired speed and the desired distance.

In the second phase, the controller of $i = 1$ does not know the correct value of u_0. Also, the macroscopic variable is not available to it: for these reasons, it does not succeed to perfectly track the desired distance neither in case of positive disturbance between $t = 10s$ to $t = 15s$ nor in case of negative one between $t = 15s$ to $t = 20s$ (see Figure 2). However, it converges to the same speed of $i = 0$ after a small transient of three seconds in both cases (see Figure 3). Finally, at $t = 20s$ the disturbance is not active anymore and the leader can restore its desired speed. Also, $i = 1$ receives correct information about its leader acceleration and is able to return to the ideal distance. The dynamical evolution of the remaining vehicles in the platoon during the generated transients after $t = 10s$, $t = 15s$ and $t = 20s$ catches the contribution of the macroscopic information. To this purpose, let us consider the speed dynamics of the last vehicle in Figure 3. It is possible to remark an anticipatory behaviour due to the macroscopic information resulting in a higher speed between $t = 10s$ and $t = 11s$ with respect to the leading vehicles. The decreasing of vehicles' speed along the platoon scales with respect to their position, resulting more stressed in the last vehicles (see between $t = 12s$ and $t = 13s$). The same anticipatory behaviour is shown in Figure 4 with respect to the accelerations of the leading vehicles. The acceleration profiles better show

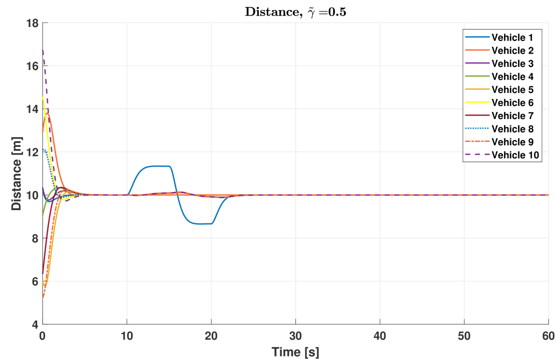

Fig. 2. Control strategy for variable spacing policy: Distances.

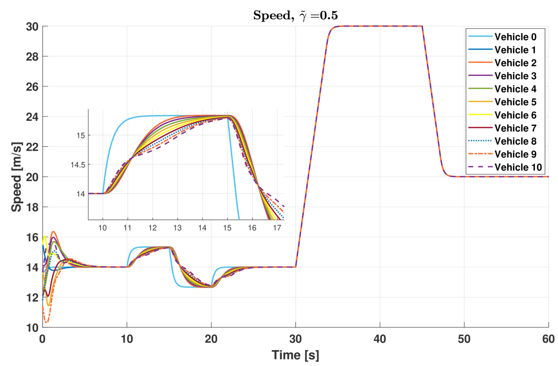

Fig. 3. Control strategy for variable spacing policy: Speeds.

how the vehicles along the platoon scale to intensify their accelerations and speeds, both for increasing and decreasing speed phases. An anticipatory behaviour is shown, either when the leading vehicles are accelerating or converging to the same speed. The same applies for transients taking place after $t = 15s$ and $t = 20s$, which are generated by the fast reaction of the leading vehicle to the disturbance. An anticipatory harmonizing acceleration for each vehicle scales along the platoon with respect to their knowledge of the macroscopic quantities, as clearly shown in Figure 3 and 4.

In the third phase, since there is no unknown perturbation acting on the platoon, the vehicles succeed to track the variable speed profile and to remain at the desired distance. No oscillations are shown by the proposed control law, even if the desired speed profile has high steps.

The proposed control laws in (27) exploit the information resulting from the macroscopic variable and safely control a platoon of vehicles. The control inputs provide transient harmonization on the whole traffic throughput while ensuring Asymptotic String Stability properties. The dynamical evolution results in a reduction of the oscillations propagation along the platoon, both in nominal case and with an active external disturbance. The utilization of the macroscopic information results to be a powerful tool.

VI. Conclusions

This paper introduces macroscopic variables for improving String Stability performance of a platoon of CACC au-

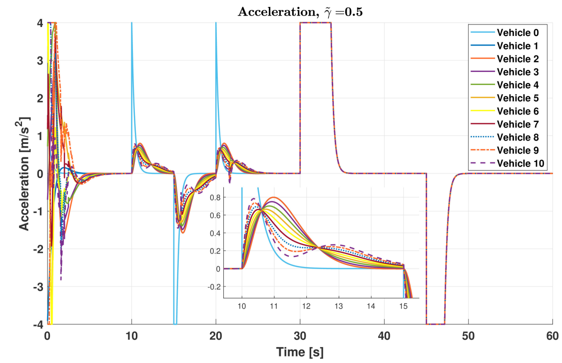

Fig. 4. Control strategy for variable spacing policy: Accelerations.

tonomous vehicles. As the variance of microscopic quantities is related to the macroscopic density, the proposed stability analysis opens to the possibility to properly control a platoon by propagating only macroscopic density information. A control law based on information obtained by V2V communication has been proposed. The improvements resulting from taking into account macroscopic information are shown by simulation results. The proposed mesoscopic control law produce an anticipatory behaviour, which provides a better transient harmonization. Future work will focus on extending the proposed framework in a mixed traffic situation with non-communicating vehicles.

References

[1] E. Uhlemann, "The us and europe advances v2v deployment [connected vehicles]," *IEEE Vehicular Technology Magazine*, vol. 12, no. 2, pp. 18–22, 2017.

[2] S. Feng, Y. Zhang, S. E. Li, Z. Cao, H. X. Liu, and L. Li, "String stability for vehicular platoon control: Definitions and analysis methods," *Annual Reviews in Control*, vol. 47, pp. 81–97, 2019.

[3] B. Besselink and K. H. Johansson, "String stability and a delay-based spacing policy for vehicle platoons subject to disturbances," *IEEE Transactions on Control Systems Technology*, vol. 62, no. 9, pp. 4376–4391, 2017.

[4] A. Alam, B. Besselink, V. Turri, J. MåRtensson, and K. H. Johansson, "Heavy-duty vehicle platooning for sustainable freight transportation: A cooperative method to enhance safety and efficiency," *IEEE Control Systems Magazine*, vol. 35, no. 6, pp. 34–56, 2015.

[5] T. Nagatani, "Traffic jams induced by fluctuation of a leading car," *Phys. Rev. E*, vol. 61, pp. 3534–3540, 2000.

[6] D. Swaroop and J. K. Hedrick, "String stability of interconnected systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 349–357, 1996.

[7] R. E. Stern, S. Cui, M. L. Delle Monache, R. Bhadani, M. Bunting, M. Churchill, N. Hamilton, R. Haulcy, H. Pohlmann, F. Wu, B. Piccoli, B. Seibold, J. Sprinkle, and D. B. Work, "Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments," *Transportation Research Part C: Emerging Technologies*, vol. 89, pp. 205 – 221, 2018.

[8] P. Caravani, E. De Santis, F. Graziosi, and E. Panizzi, "Communication control and driving assistance to a platoon of vehicles in heavy traffic and scarce visibility," *Intelligent Transportation Systems, IEEE Transactions on*, vol. 7, no. 4, pp. 448–460, 2006.

[9] M. Treiber and A. Kesting, *Traffic flows dynamics*. Springer, 2013.

[10] Z. He, L. Zheng, L. Song, and N. Zhu, "A jam-absorption driving strategy for mitigating traffic oscillations," *IEEE Transactions on Intelligent Transportation Systems*, vol. 18, no. 4, pp. 802–813, 2017.

[11] G. Piacentini, M. Čičić, A. Ferrara, and K. H. Johansson, "VACS equipped vehicles for congestion dissipation in multi-class CTM framework," in *2019 18th European Control Conference (ECC)*, pp. 2203–2208, 2019.

[12] C. Pasquale, S. Sacone, S. Siri, and A. Ferrara, "A new Micro-Macro METANET model for platoon control in freeway traffic networks," in *2018 21st International Conference on Intelligent Transportation Systems (ITSC)*, pp. 1481–1486, 2018.

[13] A. Jamshidnejad, I. Papamichail, M. Papageorgiou, and B. De Schutter, "A mesoscopic integrated urban traffic flow-emission model," *Transportation Research Part C: Emerging Technologies*, vol. 75, pp. 45 – 83, 2017.

[14] A. Iovine, F. Valentini, E. De Santis, M. D. Di Benedetto, and M. Pratesi, "Safe human-inspired mesoscopic hybrid automaton for autonomous vehicles," *Nonlinear Analysis: Hybrid Systems*, vol. 25, pp. 192 – 210, 2017.

[15] D. Yanakiev and I. Kanellakopoulos, "Variable time headway for string stability of automated heavy-duty vehicles," in *34th IEEE Conference on Decision and Control*, (New Orleans, LA), 1995.

[16] J. Zhang and P. Ioannou, "Adaptive vehicle following control system with variable time headways," in *44th IEEE Conference on Decision and Control*, (Seville, Spain), 2005.

[17] H. K. Khalil, *Nonlinear systems*. Prentice Hall, 2002.

[18] J. Tian, H. Zhang, M. Treiber, R. Jiang, Z.-Y. Gao, and B. Jia, "On the role of speed adaptation and spacing indifference in traffic instability: Evidence from car-following experiments and its stochastic model," *Transportation Research Part B: Methodological*, vol. 129, pp. 334 – 350, 2019.

[19] L. Xiao and F. Gao, "Practical string stability of platoon of adaptive cruise control vehicles," *Intelligent Transportation Systems, IEEE Transactions on*, vol. 12, no. 4, pp. 1184–1194, 2011.

[20] J. Ploeg, N. van de Wouw, and H. Nijmeijer, "$\mathcal{L}_p$ string stability of cascaded systems application to vehicle platooning," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 786–793, 2014.

[21] Y. Zheng, S. Ebel Li, K. Li, F. Borrelli, and J. Hedrick, "Distributed model predictive control for heterogeneous vehicle platoons under unidirectional topologies," *IEEE Transactions on Control Systems Technology*, vol. 25, no. 3, pp. 899–910, 2017.

[22] Y. Zheng, S. Eben Li, J. Wang, L. Wang, and K. Li, "Influence of information flow topology on closed-loop stability of vehicle platoon with rigid formation," in *17th International IEEE Conference on Intelligent Transportation Systems (ITSC)*, 2014.

[23] D. Swaroop and J. Hedrick, "Constant spacing strategies for platooning in automated highway systems," *Journal of Dynamic Systems Measurement and Control*, vol. 121, no. 3, 1999.

[24] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1835–1842, 2004.

APPENDIX

A. Calculations of $\underline{\alpha}$ and $\overline{\alpha}$ for Lemma 1

Here we exploit the calculations that show how (34) satisfies (35) and (36). Indeed,

$$\begin{aligned}W(\tilde{\chi}_i) &= \frac{1}{2}(\Delta p_i + \Delta\bar{p} + \rho_{1,i-1})^2 \\ &+ \frac{1}{2}(\Delta v_i - \lambda_1\rho_{1,i-1} + \rho_{2,i-1} + K_{\Delta p}(\Delta p_i + \Delta\bar{p} + \rho_{1,i-1}))^2 \\ &+ \frac{1}{2}\rho_{1,i-1}^2 + \frac{1}{2}\rho_{2,i-1}^2 \\ &= \frac{1}{2}\tilde{\chi}_i^T \underbrace{\begin{bmatrix} 1+K_{\Delta p}^2 & p_1 & p_2 & p_1 \\ 0 & 1 & p_3 & 2 \\ 0 & 0 & 2+(\lambda_1 - K_{\Delta p})^2 & p_3 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{P_W} \tilde{\chi}_i\end{aligned}$$

where

$$p_1 = 2K_{\Delta p},\ p_2 = 2(1 + K_{\Delta p}^2 - \lambda_1 K_{\Delta p}),$$
$$p_3 = 2(K_{\Delta p} - \lambda_1).$$

By defining $\lambda_{\min}(P), \lambda_{\max}(P)$ respectively the minimum and maximum eigenvalues of the generic matrix P, then,

$$\frac{1}{2}\lambda_{\min}(P_W) \leq W(\tilde{\chi}_i) \leq \frac{1}{2}\lambda_{\max}(P_W).$$

Since $K_{\Delta p}, \lambda_1, \lambda_2 > 0$, then

$$\lambda_{\min}(P_W) = 1,\ \lambda_{\max}(P_W) = \max\{1+K_{\Delta p}^2, 2+(\lambda_1 - K_{\Delta p})^2\}.$$

Consequently, (35) and (36) are verified.

B. Calculations of α for Lemma 1

Here we exploit the calculations in (37), and define α in (38).

$$\begin{aligned}\dot{W}(\tilde{\chi}_i) &= -K_{\Delta p}(\Delta p_i + \Delta\bar{p} + \rho_{1,i-1})^2 - \lambda_1\rho_{1,i-1}^2 - \lambda_2\rho_{2,i-1}^2 \\ &- K_{\Delta v}(\Delta v_i - \lambda_1\rho_{1,i-1} + \rho_{2,i-1} + K_{\Delta p}(\Delta p_i + \Delta\bar{p} + \rho_{1,i-1}))^2 \\ &+ \rho_{1,i-1}\rho_{2,i-1} + \rho_{2,i-1}(a\psi_{\Delta p}^{i-1} + b\psi_{\Delta v}^{i-1}) \\ &= -\tilde{\chi}_i^T \underbrace{\begin{bmatrix} q_1 & 2K_{\Delta p}K_{\Delta v} & q_2 & 2K_{\Delta p}K_{\Delta v} \\ 0 & K_{\Delta v} & q_3 & 2K_{\Delta v} \\ 0 & 0 & q_4 & q_5 \\ 0 & 0 & 0 & \lambda_2 + K_{\Delta v} \end{bmatrix}}_{Q_W} \tilde{\chi}_i \\ &+ |\tilde{\chi}_i|(a|\psi_{\Delta p}^{i-1}| + b|\psi_{\Delta v}^{i-1}|) \\ &\leq -\lambda_{\min}(Q_W)|\tilde{\chi}_i|^2 + |\tilde{\chi}_i|(a|\psi_{\Delta p}^{i-1}| + b|\psi_{\Delta v}^{i-1}|)\end{aligned}$$

where $\lambda_{\min}(Q_W) = \min\{q_1, K_{\Delta v}, q_4, \lambda_2 + K_{\Delta v}\}$ and

$$\begin{aligned}q_1 &= K_{\Delta p}(1 + K_{\Delta p}K_{\Delta v}), \\ q_2 &= 2K_{\Delta p}(1 + K_{\Delta v}(K_{\Delta p} - \lambda_1)), \\ q_3 &= 2K_{\Delta v}(K_{\Delta p} - \lambda_1), \\ q_4 &= K_{\Delta p} + \lambda_1 + K_{\Delta v}(\lambda_1 - K_{\Delta p})^2, \\ q_5 &= 1 - 2K_{\Delta v}(K_{\Delta p} - \lambda_1).\end{aligned}$$

Consequently the inequality in (37) is verified, with $\alpha = \lambda_{\min}(Q_W)$.

C. Macroscopic functions inequalities for Lemma 1

First we recall the following variance property: let $l \in \{1, ..., m\}$ and $y_l \in \mathbb{R}$, then the variance with respect to the set of values y_l, σ_y^2, is such that

$$\sigma_y^2 \leq \frac{1}{4}(\max_l y_l - \min_l y_l)^2$$

We consider the dynamics in (25) and (26) with respect to $\tilde{\chi}_i = \hat{\chi}_i - \chi_{e,i}$. Let us define $\Delta\tilde{p}_i = \Delta p_i - \Delta\bar{p}$, then for the macroscopic functions $\psi_{\Delta p}^i$ and $\psi_{\Delta v}^i$, the following inequality is proved:

$$\begin{aligned}|\psi_{\Delta p}^i| &\leq \gamma_{\Delta p}\sqrt{\sigma_{\Delta p}^2} \\ &\leq \frac{1}{2}\gamma_{\Delta p}|\max_{j=0,\dots,i}\Delta\tilde{p}_j - \min_{j=0,\dots,i}\Delta\tilde{p}_j| \\ &\leq \gamma_{\Delta p}\max_{j=0,\dots,i}|\Delta\tilde{p}_j| \\ &= \gamma_{\Delta p}\max_{j=0,\dots,i}\sqrt{(\Delta\tilde{p}_j)^2 + 0\cdot(\Delta v_j)^2 + 0\cdot(\rho_{j-1})^2} \\ &= \gamma_{\Delta p}\max_{j=0,\dots,i}|\tilde{\chi}_j|\end{aligned}$$

where we have exploited the relationship

$$|\max_j \Delta\tilde{p}_j| \leq \max_j |\Delta\tilde{p}_j|$$
$$|\min_j \Delta\tilde{p}_j| \leq \max_j |\Delta\tilde{p}_j|$$

By applying the same methodology, is proven the inequality

$$|\psi_{\Delta v}^i| \leq \gamma_{\Delta v}\max_{j=0,\dots,i}|\tilde{\chi}_j|$$

Then

$$a|\psi_{\Delta p}^i| + b|\psi_{\Delta v}^i| \leq (a\gamma_{\Delta p} + b\gamma_{\Delta v})\max_{j=0,\dots,i}|\tilde{\chi}_j|$$

D. Macroscopic functions inequalities for Corollary 1

We consider the dynamics in (25) and (26) with respect to $\tilde{\chi}_i = \hat{\chi}_i - \chi_{e,i}$. Let us define $\Delta\tilde{p}_i = \Delta p_i - \Delta\bar{p}$, then for the macroscopic functions $\psi_{\Delta p}^i$ and $\psi_{\Delta v}^i$, the following inequality is proved:

$$\begin{aligned}|\psi_{\Delta p}^i| &\leq \gamma_{\Delta p}\sqrt{\sigma_{\Delta p,i}^2} \\ &= \gamma_{\Delta p}\left(\frac{1}{i+1}\sum_{j=0}^{i}\Delta\tilde{p}_j^2 - \frac{1}{(i+1^2)}\left(\sum_{j=0}^{i}\Delta\tilde{p}_j\right)^2\right)^{\frac{1}{2}} \\ &\leq \gamma_{\Delta p}\frac{1}{\sqrt{i+1}}\left(\sum_{j=0}^{i}\Delta\tilde{p}_j^2\right)^{\frac{1}{2}} \\ &\leq \gamma_{\Delta p}\frac{1}{\sqrt{i+1}}\sum_{j=0}^{i}|\Delta\tilde{p}_j|\end{aligned}$$

where we have used the inequality $|x|_2 \leq |x|_1$. In the same way we can prove that

$$|\psi_{\Delta v}^i| \leq \gamma_{\Delta v}\frac{1}{\sqrt{i+1}}\sum_{j=0}^{i}|\Delta v_j|$$

Then,

$$
\begin{aligned}
a|\psi^i_{\Delta p}| + b|\psi^i_{\Delta v}| &\leq \frac{1}{\sqrt{i+1}} \left(a\gamma_{\Delta p} \sum_{j=0}^{i} |\Delta \tilde{p}_j| + b\gamma_{\Delta v} \sum_{j=0}^{i} |\Delta v_j| \right) \\
&= \frac{1}{\sqrt{i+1}} \sum_{j=0}^{i} \left| \left[\begin{array}{ccc} a\gamma_{\Delta p} & 0 & 0 \\ 0 & b\gamma_{\Delta v} & 0 \\ 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} \Delta \tilde{p}_j \\ \Delta v_j \\ \rho_{j-1} \end{array} \right] \right|_1 \\
&\leq \sqrt{\frac{3}{i+1}} \max\{a\gamma_{\Delta p}, b\gamma_{\Delta v}\} \sum_{j=0}^{i} |\tilde{\chi}_j|
\end{aligned}
$$