Guided Project 36: Predator-prey models

Topics and skills: Integration

The remarkable graph in Figure 1 shows nearly 100 years of data of hare and lynx populations (collected south of Hudson Bay in Canada). Both populations show distinct cycles with periods of approximately 12 years that are slightly out of phase (the peaks of the two curves are offset by a few years). Wildlife ecologists theorize that the hare and lynx populations displayed in this graph must have interacted in a special way to produce these regular cycles. Specifically, lynx (the predator) fed on hares (the prey) until the hare population was reduced to low levels. With a shortage of food, the lynx began to die. However, the absence of lynx allowed the hare population to recover, which in turn increased the food supply for lynx. Such predator-prey cycles are exhibited clearly in these data. Our goal is to devise a simple mathematical model that shows the same kind of oscillation. (Adapted from Odum, Fundamentals of Ecology, Saunders, 1953)

Figure 1

1. The model is based on two assumptions:
   A1: In the absence of lynx, the hare population increases exponentially. However, the hare population decreases in proportion to encounters between hares and lynx.
   A2: In the absence of hares, the lynx population decreases exponentially. However, the lynx population increases in proportion to encounters between hares and lynx.
   We let \( H(t) \) and \( L(t) \) denote the lynx and hare populations, respectively, at time \( t \geq 0 \). Consider the following differential equations that give the rates of change of \( H \) and \( L \).

\[
\frac{dH}{dt} = \frac{aH}{\text{natural growth rate}} - \frac{bHL}{\text{hare-lynx interactions}}
\]

\[
\frac{dL}{dt} = -\frac{cL}{\text{natural decay rate}} + \frac{dHL}{\text{hare-lynx interactions}}
\]

where \( a, b, c, \) and \( d \) are positive constants. Explain how the terms \( aH \) and \( -bHL \) in the first differential equation reflect assumption A1.

2. Explain how the terms \( -cL \) and \( dHL \) in the second differential equation reflect assumption A2.
3. The differential equations given above involve two unknown functions, \(H\) and \(L\). So we need a useful way to display both solutions at once. Think of \(H(t)\) and \(L(t)\) as functions of a parameter \(t\), which together describe a parametric curve in the \(HL\)-plane (this plane is called the phase plane). The initial populations \((H(0), L(0))\) correspond to a point in the plane. As \(t\) increases, a curve is generated consisting of the points \((H(t), L(t))\) (Figure 2). Question: If \(H\) and \(L\) vary cyclically, as in Figure 1, what would be the general shape of the solution curve in the \(HL\)-plane?

4. Let’s now use specific values of the coefficients; consider the equations

\[
\begin{align*}
dH &= 0.4H - 0.02HL \\
dL &= -0.3L + 0.005HL.
\end{align*}
\]

It turns out that these equations cannot be solved explicitly for \(H\) and \(L\). Instead we will do some graphical analysis that tells us a lot about the solutions. Because \(H\) and \(L\) are populations, they are positive and we consider only the first quadrant of the \(HL\)-plane.

An equilibrium in the system occurs when

\[
\frac{dH}{dt} = \frac{dL}{dt} = 0
\]

(neither population changes in time). Solve two algebraic equations to show that the equilibrium states of the system are \(H = L = 0\), which is not very interesting, and \(H = 60, L = 20\). The equilibrium point is shown in Figure 3 as the intersection of the lines \(H = 60\) and \(L = 20\).

5. We now work in several steps to produce a direction field in the \(HL\)-plane that shows the general shape of the solution curves. Let’s start with the first equation in (1). Show that \(H'(t) = 0\) when \(L = 20\). This means that if a solution curve crosses the line \(L = 20\), the hare population is not changing, so the solution curve is vertical along \(L = 20\). We indicate this fact by putting small vertical line segments on \(L = 20\) (Figure 3).

6. Use the first equation in (1) to show that

- \(\frac{dH}{dt} > 0\) if \(0 < L < 20\)
- \(\frac{dH}{dt} < 0\) if \(L > 20\)
7. Explain why the observations of Step 6 are reflected in the \( HL \)-plane using right-arrows (\( \rightarrow \) ) and left-arrows (\( \leftarrow \) ) in Figure 4.

8. Using the second equation in (1), argue as in Step 5 and show that if \( H = 60 \), then \( \frac{dL}{dt} = 0 \). Explain why we put small horizontal line segments along the line \( H = 60 \).

9. Use the second equation in (1) show that \( \frac{dL}{dt} > 0 \) if \( H > 60 \) and \( \frac{dL}{dt} < 0 \) if \( 0 < H < 60 \).

10. Explain why the observations of Step 9 are reflected in the \( HL \)-plane using up-arrows (\( \uparrow \) ) and down-arrows (\( \downarrow \) ) in Figure 4.

11. In each of the four regions in Figure 4, look at the combined effect of the arrows to find the overall direction of the solution curves. For example, in the region \( 0 < H < 60, \ 0 < L < 20 \), the solutions move in the negative \( L \)-direction and the positive \( H \)-direction. Do a similar analysis in the other three regions. Conclude that the solutions move in a counterclockwise direction around the equilibrium point (Figure 5).
Extra Pt. Assignment for 3 – 5 Points

In completing the provided Assignment for DE Diff. Equations.

Points will be awarded as follows:

- To earn 3 Pts. = you must complete 70%-79% accurately
- To earn 4 Pts. = you must complete 80%-89% accurately
- To earn 5 Pts. = you must complete 90%-100% accurately

You will need to complete your work in your own handwriting and scan to Ms. Y Cui. cuiy@scsk12.org no later than 11:59 p.m. on May 11. 2020. Work received after this deadline will not be eligible for additional points.

If you have any questions about your assignment, please email Ms. Y. Cui. cuiy@scsk12