An exploration for the best-suit musical scale with a mathematical approach

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Abstract

This study examines three major musical scales - the Pythagorean Scale, Just Intonation, and Equal Temperament - to determine which provides the optimal balance of harmonic quality and practical utility. The analysis focuses on two key criteria: the ability to produce harmonious chords (evaluated through frequency ratios) and capacity for perfect transposition (assessed by consistency of adjacent note intervals). Through mathematical analysis and experimental validation via listener surveys, the research finds that while Just Intonation produces the most harmonious chords due to its simple frequency ratios, Equal Temperament emerges as the most practical scale overall due to its perfect transposition capabilities. The slight reduction in chord harmony in Equal Temperament proves negligible in practice, making it the most suitable choice particularly for fixed-pitch instruments like pianos. The findings help explain why Equal Temperament has become the dominant tuning system in modern Western music while acknowledging the situational benefits of other scales for specific applications.

Keywords: Music, Harmonious chords, Mathematical ratios, Harmony, Frequency

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Introduction

Musical scale(s) as the fundamental part of music has math behind to explain its pattern. (simplifying theory, 2024) We hear pleasant and harmonic sounds, while math can explain the reason. Though in the realm of music, one's subjective feeling and emotion is greatly valued, Math helps to demonstrate the pattern behind with objective evidence and can even provide suggestions for improvement. Thus, I am inspired to use mathematical approaches to explore the best-fit musical scale in making the most harmonious music.

Background information

2.1 Basic information for Musical scales

Vibrations create soundwaves, which generate sound. (Science World, 2024) Frequency, the number of times a soundwave repeats itself in a second, has a unit of Hz, and determines the pitch of the sound, that is, whether the sound sounds high or low. (gov, 2018) There can be infinite frequencies for soundwaves, and thus infinite sounds.

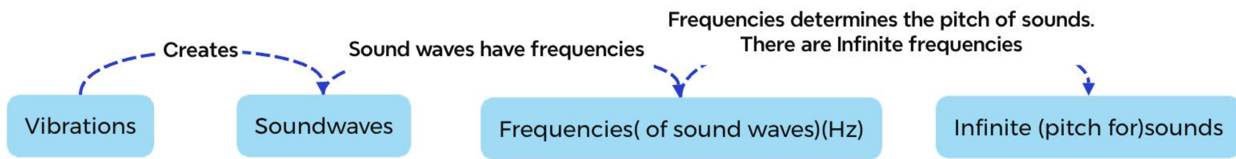

Figure 1 Relationship of vibrations, frequencies, and sounds

However, the rhythms we hear from songs and musical performances are all created by sounds chosen from "a certain set of sounds". This "set" is also known as the musical scale, and the sounds within the musical scale are called "notes". Musical scale contains 12 notes in total.

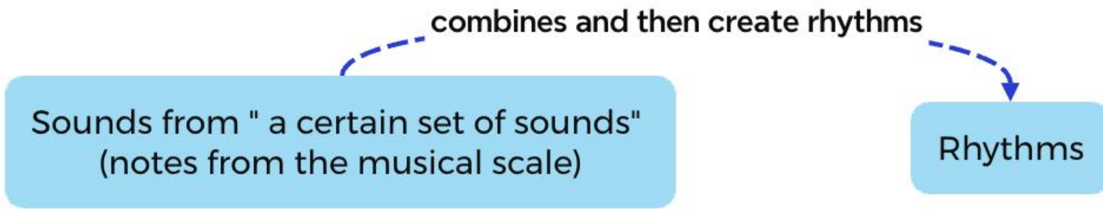

Figure 2 Notes from the musical scale combines and give rise to what is known as rhythms

In the scale, the 7 basic notes are: C, D, E, F, G, A, B, (Do Re Mi Fa Sol La Ti Do) and the other 5 notes are the sharps and flats that are named based on the basic notes. A set of the 12 notes is one octave for the musical scale, meaning that the 13th note would be back to C again. The 13th note (8th basic note) C is not the exact same note with the 1st note C, but they are in an octave relationship. If the frequencies of two notes have a ratio of $\frac{1}{2}$, they are in an octave relation. (DeVoto, 2024) When two notes are in an octave relation, they sound like the same note but in a different pitch. Besides the $\frac{1}{2}$ ratio of two notes, an octave is also defined as the set of all the notes in between the two notes with the $\frac{1}{2}$ ratio. For example, the middle C (the C closest to the center of the piano) has a frequency of about 261.63 Hz, and with rising sequence the next C (call it C') is about 523.25Hz. (Byrd, 2007) Hence, the middle C and C' has a relation of an octave, and the set of all the notes within middle C and C' is also called an octave. An octave is another way to call the scale, since the scale is the set of notes with which

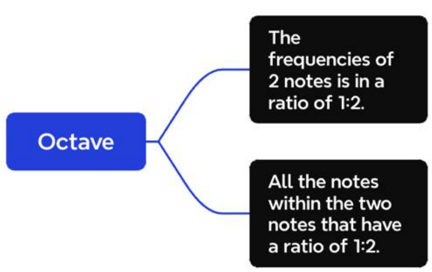

Figure 3 Two definitions of Octave

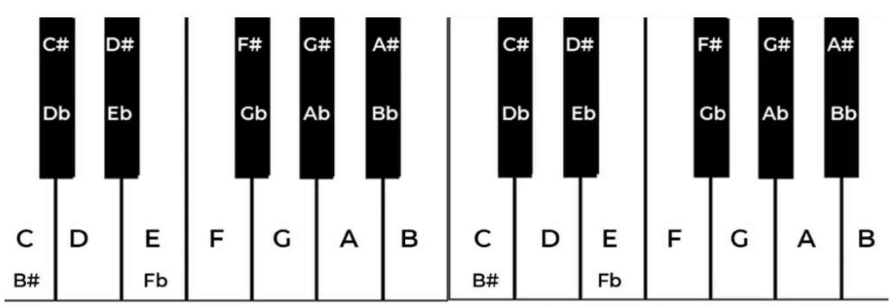

Figure 4 Image of the names of notes on musical scale, and description of octave (Laghavi, 2024)

its 1st and the 13th (8th basic) note has a $\frac{1}{2}$ (frequency) ratio. The scale/octave can be more clearly demonstrated on the keyboards of pianos.

Rather than a fixed set of certain frequencies(sounds), the musical scale is more like a rule for the "ratio" of notes. The "rule" is not only that the scale's 1st and 13th (8th basic) note should have a (frequency) ratio of $\frac{1}{2}$, but also rules for the ratio between the notes that are within the scale.

Musical scale is always 12 notes (13 if the 13th C is included) in total and 7 basic notes (C D E F G A B) (8 if the 8th basic note C is included. The naming of notes is unchanging, and the octave rule that a musical scale is an octave is widely accepted and has become a definition of "musical scale". However, the rules for the notes within the scale are not universal. Different rules result in different scales (called differently). There could be infinite types of musical scales as anyone could assign any rules of ratio to their scale and give it a name. Nevertheless, the best musical scale should be most capable in providing a good auditory experience, which has its math behind.

There are about a hundred used and recognized musical scales throughout history. (EarMaster, 2024) In the essay, I will analyze and find out the best-suit one to be used among the 3 most popular musical scales: The Pythagorean tunning, Just Intonation, and The Equal Temperament. The Pythagorean tuning is of great significance as the first creation of musical scales and offers insights on the math between creating a scale, while the other two are the most popularly used ones nowadays. (EarMaster, 2024)

2.2 Features of a good Musical scale

In the essay, I will analyze, with mathematical approaches, the capabilities of the three musical scales in providing the best auditory experience. Nonetheless, what are the criteria for a good musical scale? Though there are not strict criteria defined, from reviews commenting on the quality of musical scales, I summarized two main features of a good scale and will base my examination on them.

1) a good musical scale creates harmonious chords. A chord is when two or more notes happen at the same time. Usually, the simpler the (frequency) ratio of the notes, the more harmonious the chord is.

2) A musical scale should at best achieve perfect transposition. Transposition can happen when there is a rhythm, which is what musical scales are made for. Transposition is an act of changing the (pitch of) notes but not the relationship, which is the frequency ratio, between the notes. (Lohman, 2023) For example, a simple rhythm may be C (Dol) for 1s, and then D (Re) for 1s, and then E (Mi) for 1s. To transpose the rhythm, I need to change the pitch of the notes but not the relationship (frequency ratio) between them. If, regarding frequency ratio of C, D, E:

$$\frac{C}{D} = x, \frac{D}{E} = y$$

Then for the new transposed rhythm to be perfectly transposed,

$$\frac{the\ first\ note}{the\ second\ note} = x, \frac{the\ second\ note}{the\ third\ note} = y$$

Another practical example could be changing the pitch of a rhythm from 440Hz, 660Hz and 880Hz to 200Hz, 300Hz, and 400Hz.

Transposition is really important for a musical scale, as it determines a scale's flexibility to perform rhythms in different pitches. For example, transposition is required to change the pitch when an original rhythm doesn't match a singer's voice range.

It can be inferred that, in fact, for perfect transposition to be achieved in a musical scale, all the adjacent notes in the scale should be of the same (frequency) ratio.

2.3 The Pythagorean Tunning

The Pythagorean Tunning/Scale is based on the rule of Octave and Perfect Fifth. While the octave refers to the two notes with frequency ratio $\frac{1}{2}$, Perfect Fifth refers to two notes with frequency ratio $\frac{3}{2}$. For example, middle C has a frequency of 261.63Hz. Call it as the 1st note, then the 5th (basic) notes, G (**C** D E F **G** A B), which, according to the Pythagorean scale, is 392.45Hz.

$$\frac{391.45}{261.53} \approx 1.5 \approx \frac{3}{2}$$

The Pythagorean tunning uses Octave and Perfect Fifth to give rules for the (frequency) ratio of all the notes on the scale. First, start with the basic notes C D E F G A B. C' is added to form an octave and demonstrate the octave rule more clearly. While the first note is C on the scale, its real frequency does not matter, for as mentioned before that musical scale is more a rule for frequencies instead of merely fixes frequencies. Hence, for clearer illustration of the rule of ratios, I will assign the first note C with $\frac{1}{1}$ regardless of what its real frequency is (though with information before it can be inferred that frequency for C is always 261.63× $2^a, a \in Z$), and with the octave rule C' is $\frac{2}{1}$.

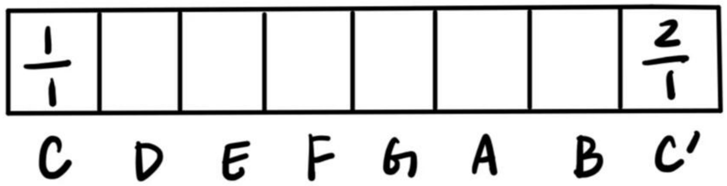

Figure 5 Octave rule shown in a scale with basic notes.

With Perfect Fifth, G is $\frac{3}{2}$. The rule of Octave and Perfect Fifth applies repeatedly in the scale. A reminder that an Octave gives a feeling of "a same note with different pitch".

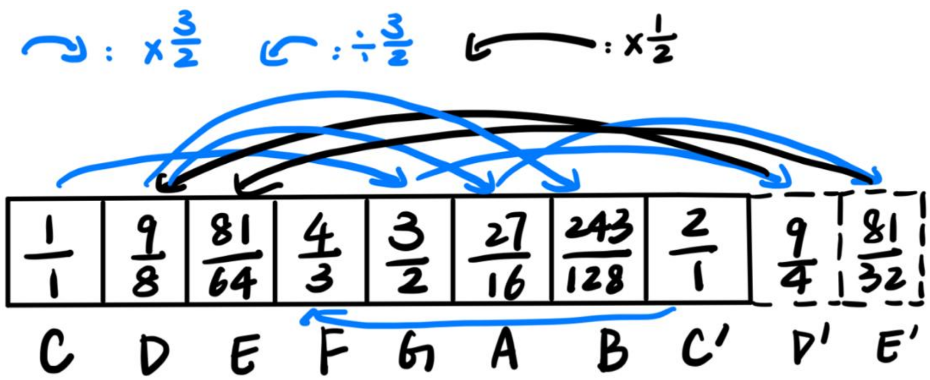

Figure 6 Construction of basic notes of the Pythagorean tunning with Perfect Fifth and Octave

As shown above, all the notes obtained is generated from the rule of Perfect fifth and Octave. These two rules with simple ratio $\frac{3}{2}$ and $\frac{1}{2}$ respectively, give the scale ability to create harmonious chords. However, to make a good musical scale, there is another criterion---transposition---which focuses on the frequency ratio of adjacent notes in the scale.

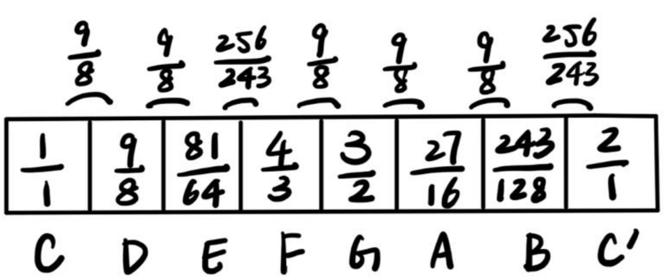

Figure 7 Frequency ratios of adjacent notes in the Pythagorean Tunning/Scale

There are inconsistencies in the frequency ratio of notes. While the smaller ratio $\frac{256}{243}$ cannot be turned into a bigger ratio $\frac{9}{8}$, $\frac{9}{8}$ can be divided smaller into $\frac{256}{243}$. Thus, 5 notes are added into the ratio $\frac{9}{8}$, which are the sharp and flats. In The Pythagorean tunning, a sharp is obtained by multiplying $\frac{256}{243}$ from the previous note, and a flat is by dividing $\frac{256}{243}$ from the 1 note ahead. For example, C is $\frac{1}{1}$, then C# is $\frac{256}{243}$; D is $\frac{2}{1}$, then Db is

$$\frac{\frac{9}{8}}{\frac{256}{243}} = \frac{2187}{2048}$$

As mentioned before, C# and Db should be the same note in a scale. The inequality of C# and Db in the Pythagorean scale thus implies a limitation of its rules which will be elaborated later. Now, I will construct a scale with the 5 notes as sharps to give a demonstration of the scale in 13 notes (including C').

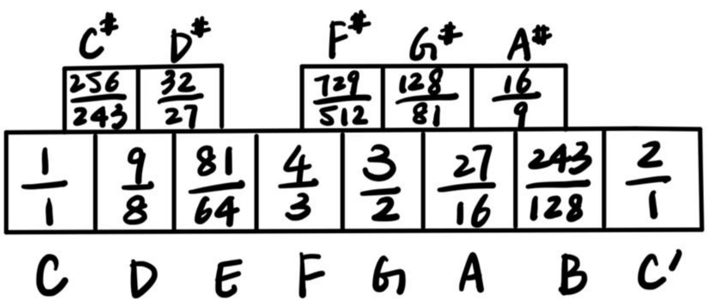

Figure 8 The Pythagorean Scale/Tunning with 13 notes (sharp notes as the 5 additional notes)

The Pythagorean Tunning can be presented geometrically with a 30-60-90 triangle. The octave can be illustrated through $\frac{2}{1}$ relationship of the hypotenuse to opposite and the Perfect Fifth can also be presented with operations.

Consider OC' as an axis, C is at 1 and C' is at 2.

$$GC' = \frac{1}{2} KC' = \frac{1}{2} \times \frac{1}{2} OC' = \frac{1}{2}$$

$$\therefore G \text{ is at } \frac{3}{2}$$

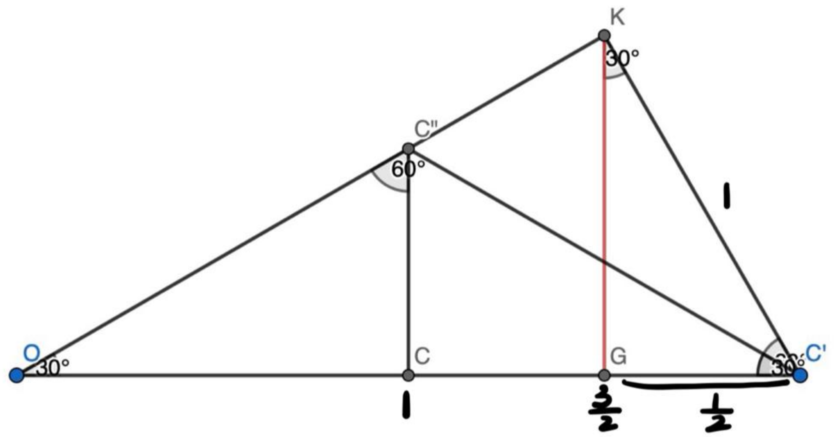

Figure 9 Demonstration of C, G, C' of Pythagorean Tunning in a 30-60-90 Triangle

$$OD = 2 - \frac{1}{2} - \frac{3}{8} = \frac{9}{8}$$

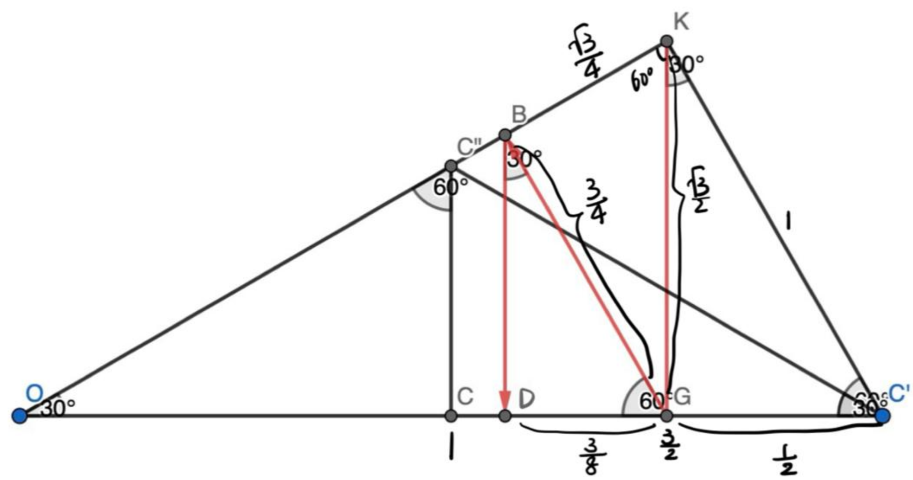

Figure 10 Demonstration of C, D, G, C' of Pythagorean Tunning in a 30-60-90 Triangle

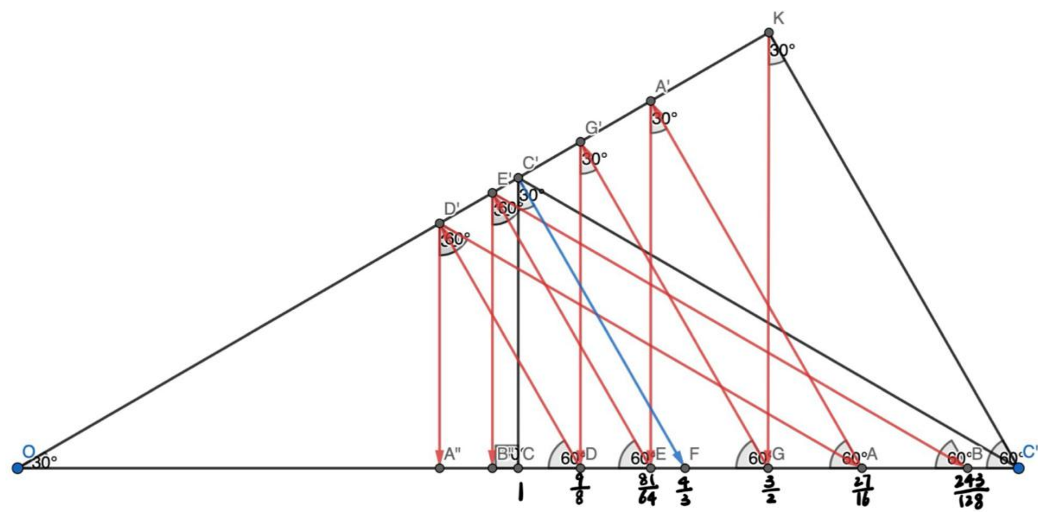

Figure 11 Demonstration of the 8 basic notes (including C') of Pythagorean Tunning in a 30-60-90 Triangle

Same with the algebraic demonstration, the 30-60-90 geometrical construct will also demonstrate the sharp notes only and leave a clearer demonstration in the analysis section.

After obtaining B as $\frac{243}{128}$,

$$\frac{OB}{OC'} = \frac{\frac{243}{128}}{2} = \frac{BB'}{C'K} = \frac{BB'}{1}$$

$$\therefore BB' = \frac{243}{256}$$

$$BF\# = BB' \times \frac{1}{2} = \frac{243}{256} \times \frac{1}{2} = \frac{243}{512}$$

$$OF\# = OC' - BC' - BF\# = 2 - (1 - \frac{243}{128}) - \frac{243}{512} = \frac{729}{512}$$

$$\therefore OF\# = \frac{729}{512}, \text{ and } F\# \text{ is at } \frac{729}{512}$$

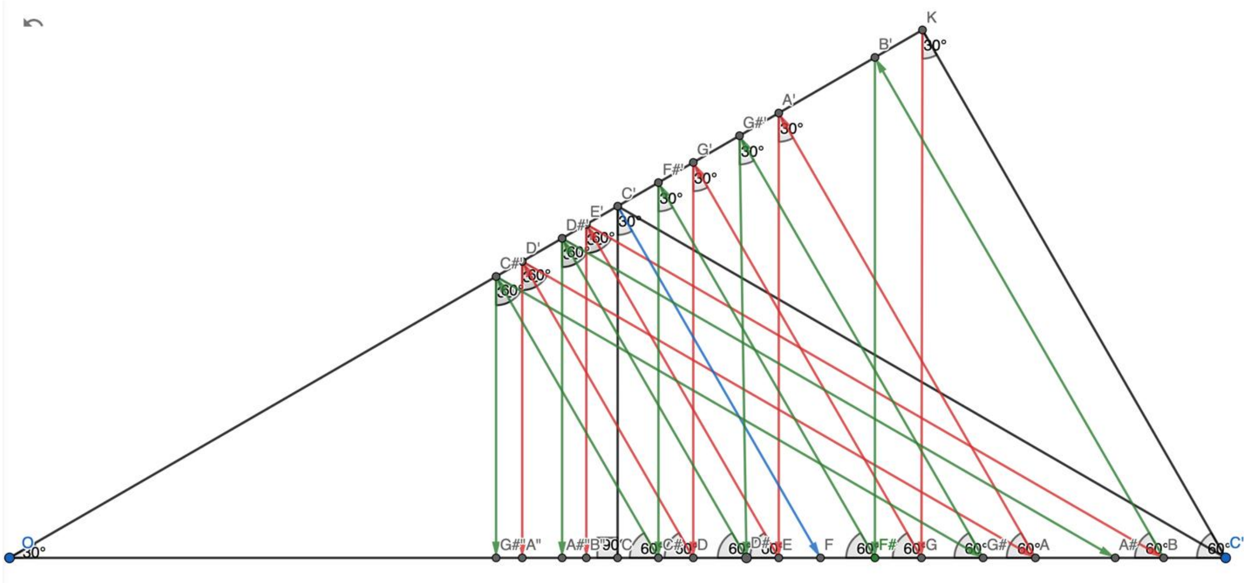

Figure 12Demonstration of the 13 notes with sharps (including C') of Pythagorean Tunning in a 30-60-90 Triangle

The numbers the notes are on refers to the table graphed earlier: (Figure 8)

C#	D#		F#	G#	A#	
256/243	32/27		729/512	128/81	16/9	

C	D	E	F	G	A	B	C'
1/1	9/8	81/64	4/3	3/2	27/16	243/128	2/1

2.4 Just Intonation

After Pythagoras gave birth to the first musical scale, other musicians followed and created their own scales. The Just intonation is tuning system that is popularly used in Western musical practice, striving to assign simpler ratios to notes to give the scale an even greater ability to create good chords. (Gann, 1997) This scale is created after making some adjustments of certain ratios of the Pythagorean Scale.

Different from the Pythagorean Scale, Just Intonation did not derive the notes strictly from Perfect Fifth of Octave, rather it arbitrarily replaced some notes (e.g

E, B) with simpler ratios than that on the Pythagorean Scale to make the chords sound more harmonious. It is largely based on the Pythagorean Scale, but certain changes are made.

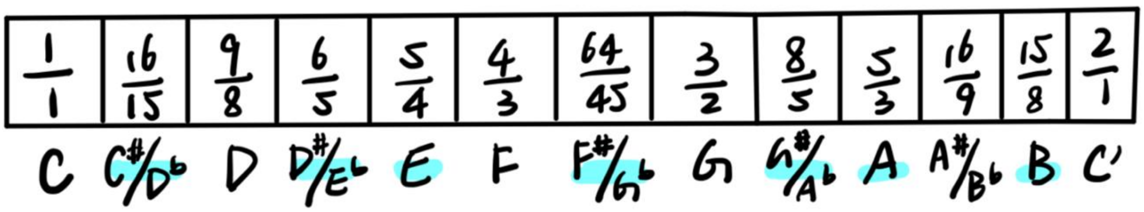

Figure 13 Full 13 notes (including C') from Just Intonation, hilighted notes as changes made from the Pythagorean scale

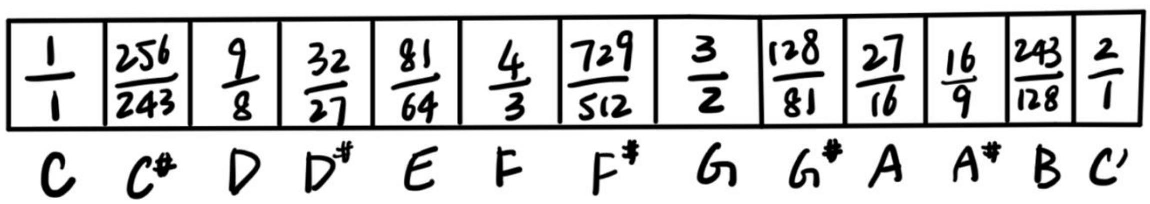

Figure 14 13 notes (including C') from the Pythagorean scale

Although for Just Intonation some ratios are arbitrarily attributed to the scale and doesn't derive from strict rules (e.g Perfect Fifth, Octave…), still, the basic notes from the scale can still be demonstrated on a 30-60-90 triangle.

In Both Just Intonation and the Pythagorean Scale, F is $\frac{4}{3}$.

$$FC' = 2 - \frac{4}{3} = \frac{2}{3}$$

$$AJC' = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

$$\therefore OAJ = 2 - \frac{1}{3} = \frac{5}{3}$$

Thus, we get the A in Just Intonation (AJ) with ratio of $\frac{5}{3}$. Repeat the operation and the notes that Just Intonation changed (E, A, B) can be presented in the triangle as well.

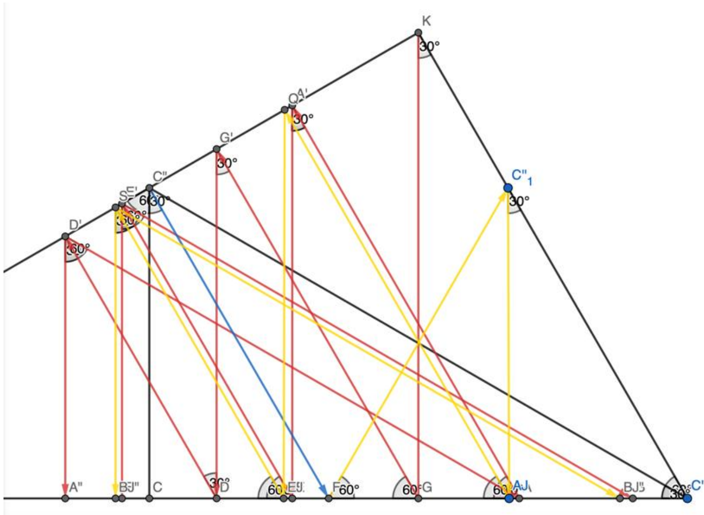

Figure 15 8 basic notes (including C') in Just Intonation, compared with the basic notes in the Pythagorean Scale

2.5 Equal Temperament

Equal Temperament is based on the principle to equally divide the notes with same interval ratios. (Formant, 2022) With 13 notes (including C') in a musical scale and in an Octave, the 12 intervals, the (frequency) ratios between all the adjacent notes, should be the same.

Figure 16 12 same Frequency ratios (presented as ratio "a") between the 12 intervals in The Equal Temperament

$$a^{12} = 2$$

$$\therefore a = \sqrt[12]{2} = 2^{\frac{1}{12}}$$

Each note in the Equal Temperament thus holds the same ratio $2^{\frac{1}{12}}$ with their adjacent notes.

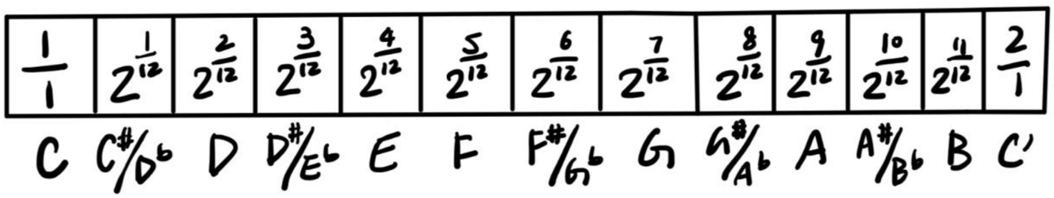

Figure 17 The Equal Temperament (13 notes, with C' included)

As shown above, since Equal temperament is defined by the strict dividing of the ratios, it conveys its feature and significance the best when presented in a linear and algebraic fashion.

Methodology

With Algebraic and geometric demonstration presented, I will then analyze the three musical scales accordingly. The analysis will be based on the 2 criteria/features mentioned earlier: 1) the harmony of chords, which the simpler the (frequency) ratio of the notes, the more harmonious the chord is. 2)Transposition, which examines the consistency of the (frequency) ratio of adjacent notes in the scale.

3.1 The analysis on Chords for the three scales.

While chords are defined as the simultaneous happening of two or more notes, it is to e understand that in most cases three-note or multiple-note chords are the result of combination of two-note chords. Thus, I will examine the performances of the three musical scale in producing harmonious chords with examples clearly presenting the ratios of the notes from a chords.

The Pythagorean Scale

$\frac{1}{1}$	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{243}{128}$	$\frac{2}{1}$
C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C'

Figure 18 13 notes (including C') from the Pythagorean scale

We do recognize the appearance of the Perfect Fifth (e.g C as 1, and G as $\frac{3}{2}$) and Octave, however we also see complicated ratios. Take the two classic two-note chords C, E and E G as an example: C: E has a ratio of $\frac{64}{81}$, and chord E: G is $\frac{27}{32}$. These two ratios seem obviously more complicated than Perfect Fifth. Nevertheless, judgements cannot be made before comparison. Thus, we look at Just Intonation with the same example.

Just Intonation

$\frac{1}{1}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{64}{45}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{2}{1}$
C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A	A#/Bb	B	C'

Figure 19 13 notes of Just Intonation

The chord C: E ratio is $\frac{4}{5}$, with a much simpler ratio than $\frac{64}{81}$ in the Pythagorean Scale. Chord E: G is $\frac{5}{6}$, comparing to $\frac{27}{32}$ in the Pythagorean Scale. Indeed, the arbitrary changes Just intonation made to the ratios of certain notes made the chords more harmonious by attributing them with simpler ratios.

Equal temperament

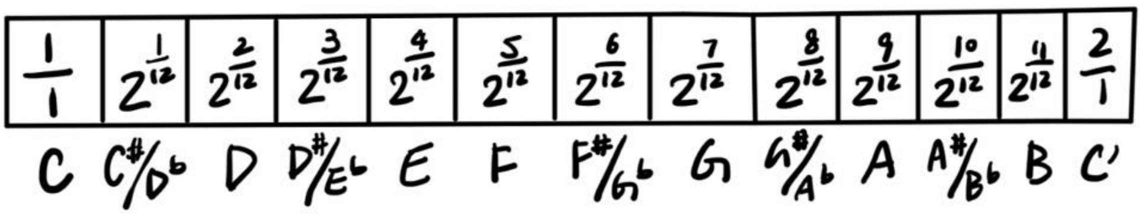

Figure 20 The Equal Temperament (13 notes, with C' included)

The equal temperament does not exhibit any ratios with integers, and not to say ratios with simple integers. The chord C: E display a ratio of $\frac{1}{2^{\frac{4}{12}}}$, and E: G is $\frac{2^{\frac{4}{12}}}{2^{\frac{7}{12}}} = \frac{1}{2^{\frac{3}{12}}}$. With the lack of whole small number ratios, the two-note chords on equal temperament sounds less harmonious compared with the other two scales. (Formant, 2022)

Till now, it seems that at least for chords of two notes, just intonation creates the most harmonious note combinations, following the Pythagorean Scale and then the equal temperament.

3.2 The analysis on transposition of the three scales.

The ability of Transposition depends on the consistency of (frequency) ratios of adjacent notes in the scale. Perfect transposition is achieved when the ratios of adjacent notes are the same.

The Pythagorean Scale

To examine its ability to transpose it is worth to look at the ratios of each of the notes to their adjacent ones.

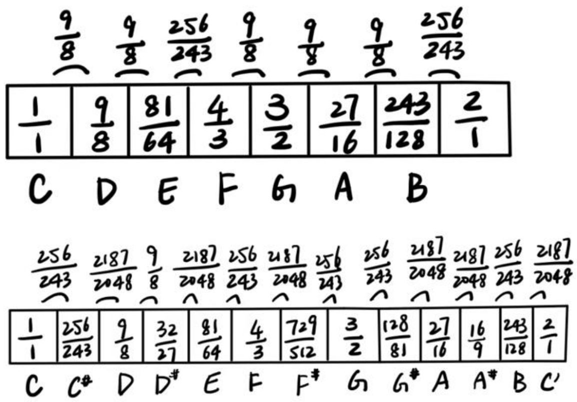

Figure 21 (frequency) ratios of adjacent notes in The Pythagorean scale (8 notes and 13 notes full scale)

To examine its ability to transpose it is worth to look at the ratios of each of the notes to their adjacent ones. The ratios being not the same in the Pythagorean scale shows that it is not able to transpose the rhythm perfectly. When transposed, the audience/human ear would notice the change in rhythm that made it less pleasant.

The scale's limitation in transposition can also be shown by the scale's inability to make the scale self-consistent by eventually returning to the starting note, which is C, $\frac{1}{1}$.

$Proof\ by\ Mathematical\ Constradiction:$

$n: No.\ times\ of\ the\ operation\ of\ a\ Perfect\ Fifth$

$m: No.\ times\ of\ the\ operation\ of\ an\ Octave$

$$1 \times (\frac{3}{2})^n \times (\frac{1}{2})^m = 1$$

$$(\frac{3}{2})^n \times (\frac{1}{2})^m = 1$$

$$\frac{3^n}{2^n} \times \frac{1}{2^m} = 1$$

$$\frac{3^n}{2^n} = 2^m$$

$$3^n = 2^{m+n}$$

$$\because\ 3^n\ is\ always\ an\ odd\ number,\ 2^{m+n}\ is\ always\ an\ even\ number,$$

$\therefore\ Proof\ by\ mathematical\ contradiction,\ the\ notes\ will\ never\ return\ to\ 1,\ which\ is$

The rules of Perfect Fifth and Octave from the scale does not allow the operation to end at the start point, meaning that the scale would require an infinite number of notes to make it consistent, otherwise there would be discrepancy in the adjacent ratios. While the Pythagorean scale presented only has 13 notes instead of infinity, there are thus inconsistency in the scale. One problem generated is called the Pythagorean comma, which is the difference in (frequency) ratio of the notes that should've been of the same ratio. (Hubbard, 2021)

In the Pythagorean scale, the sharp and flat notes that should've been of the same ratio is different from each other because of the inconsistency of the scale generated from the rule of Perfect Fifth and Octave. It is defined that in the scale, the sharp notes (#) are gained by going down in perfect Fifths (dividing $\frac{3}{2}$ which the process starts from B) . The flat notes (b), are gained by going upward in Perfect Fifths (multiplying $\frac{3}{2}$, starting from F. (Børre Nyhoff, 2023)

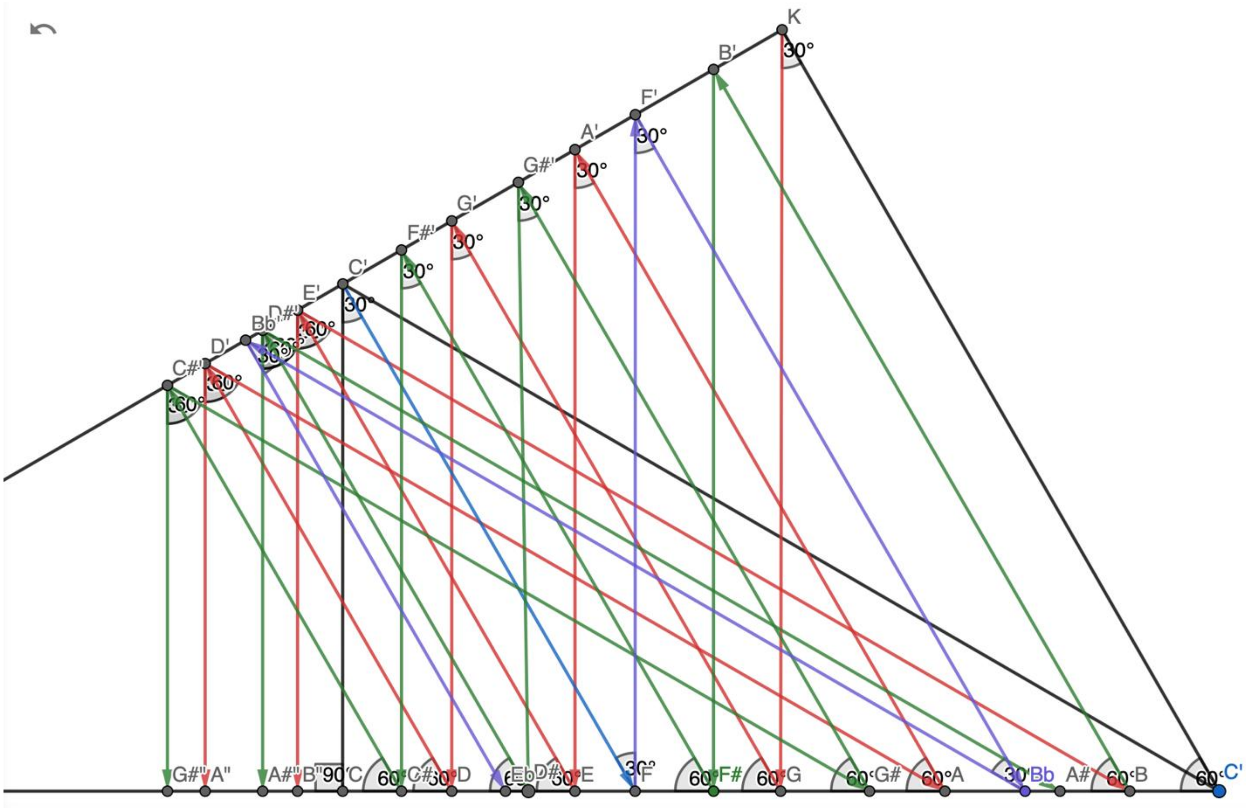

Figure 22 Geometric demonstration of the Pythagorean Comma (the divergence between flat and sharp notes that should be same)

Notice a gap/divergence between the notes that should've be the same (e.g A# and Bb, D# and Eb). This divergence is a major problem in the Pythagorean scale, as it questions us which ones of the divergent notes should we keep and which to eliminate. (Straub, 2021) Still, choosing either the flat or the sharp notes, the existence of the comma implies the inconsistency of ratios of adjacent notes in the scale: since what should've been the same notes are divergent, the scale in no way can possess all same ratios and achieve perfect transposition.

Just Intonation

In Just intonation, ratios of adjacent notes are note held the same, thus perfect transposition cannot be achieved.

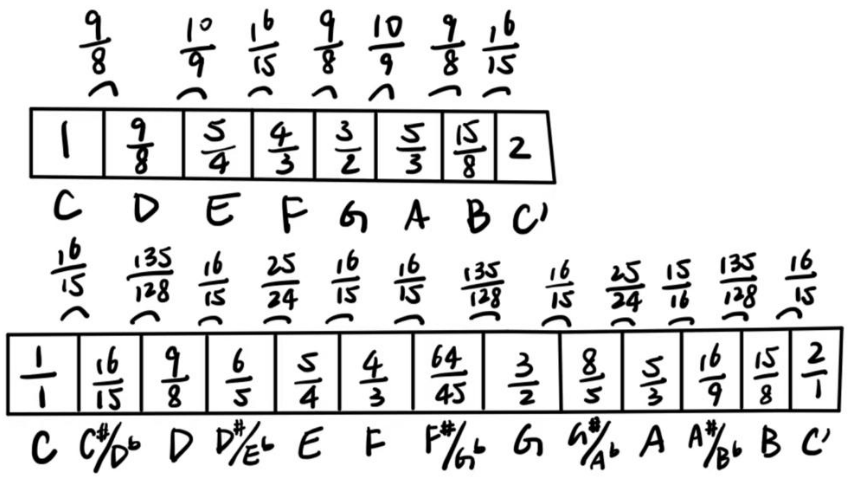

Figure 23 Note intervals of Just Intonation (7 basic notes and 13 notes full scale)

Because there are infinite rhythms in the world and every rhythm has different selection/combination of notes. Thus, the ability of transposition cannot be ranked specifically. The only thing that can be concluded from the calculations of the ratio of adjacent notes is that either two of the scales achieve perfect transposition, which as a significant part of the

The Equal Temperament

Equal temperament shows perfect transposition, an extremely crucial quality for any scale. The intentionally made equal intervals enables Equal temperament to perfectly transpose any rhythms and not twisting the auditory experience.

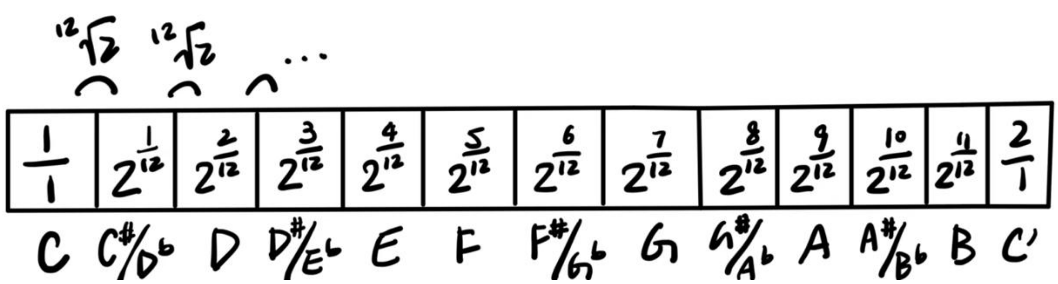

Figure 24 (adjacent) Note ratios in the Equal Temperament

While Equal temperament may not be able to produce chords as harmonious as Just intonation, its ability to perfectly transpose puts into an esteem status in the production or music and performance. (Klerk, 1979)

Though math provides objective evidence of the quality of scale, ultimately auditory experience speaks most in the quality of musical scale and justifies the mathematical evidence. Therefore, I will conduct a survey examining whether the theoretical result above align with real life auditory experience.

The survey will be composed of 2 simple questions, one as a ranking of the three scales in producing the most harmonious three-note chords, the second as a ranking of the three scales in providing the best transposition of a certain rhythm. The three scales will be given random numbers to be ranked on while there are recordings of the chords they produce and the transposition of rhythm.

Experimental Data

The chord in the recording that is played by each of the three scales is C major chord (a chord of Do, Mi, Sol), and the rhythm being played in the original scale and transposed to another key(pitch) by three scales is the Bach-Prelude in C Major (for transposition, the C major is being transposed to G major). The transposition of rhythm is done on an electric keyboard piano, switched to three of the musical scales.

37 responses of the survey are collected, with respondents ranking for the performance of the scales on the criteria of chords and transposition with recordings involved.

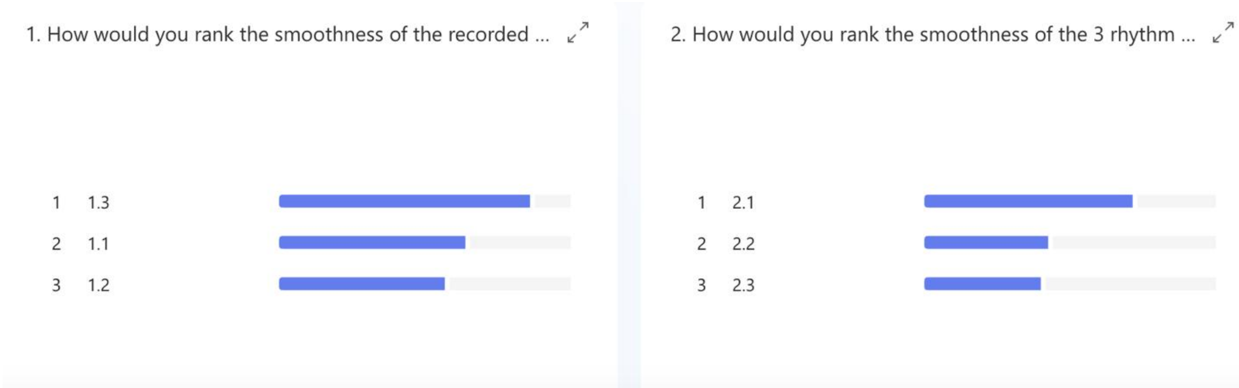

Figure 25 Data page from the respondents ranking on the three Scales' performance on chords and transposition.

The scales are being given numbers. The Equal Temperament as (1.1 and 2.1), The Pythagorean scale as (1.2 and 2.2), and Just Intonation as (1.3 and 2.3).

Hence, from the data collected, the scales are ranked:

Harmony of chords:

1.Just Intonation (1.3)

2.The Equal Temperament (1.1)

3. The Pythagorean Scale (1.2)

Harmony/quality of the rhythm Transposed:

1. The Equal Temperament (2.1)
2. The Pythagorean Scale (2.2)
3. Just Intonation (2.3)

Indeed, as what is theoretically concluded, the simple ratios resulted from the notes of the chords in Just Intonation contributes to the harmony of the chords, ranking it first by the respondents. The consistency of ratios of adjacent notes in the Equal Temperament contributed to the scale's advantage in Transposition.

Although analyzes above that the non-integer ratios in the notes of Equal Temperament would make its chord the least harmonious compared to the two scales, in the survey it is presented in the second place ahead of the Pythagorean scale. An implicature is that the quality of chords in the scales can be hard to recognize even when its being intentionally examined, and that apart from the relatively clear harmony of chords in Just Intonation, the other two scales show no big difference to human ear.

Moreover, the ability of perfect transposition, which is greatly valued for scales in fact could largely compensate for the slight roughness of chords from Equal temperament that in most cases is hard for audiences and human ear to capture and recognize and if not impossible when most of the times under formal performance no comparison of chords among other scales are taken. Therefore, The Equal Temperament is found to be the best-suit scale.

Conclusion& Evaluation

The investigation of the best fit music scale among The Pythagorean Scale, Just Intonation, and Equal Temperament concludes that Equal Temperament is the best-fit one due to its ability of transposition and ignorable flaws in chord production.

	The Pythagorean	Just Intonation	The Equal

	Scale		Temperament
Theoretical Approach analysis (on the 2 criteria)	2nd in harmony of chords. Least in transposition.	1st in Harmony of chords. Least in transposition.	3rd in harmony of chords. Achieves perfect transposition.
Experimental testing results (on the 2 criteria)	3rd in harmony of chords. 2nd in transposition.	1st in Harmony of chords. 3rd in transposition.	2nd in harmony of chords. 1st in transposition

Still, there are some limitations recognized and potential solutions in further investigations:

Limitations	Improvements
Not enough chords being analyzed theoretically or recorded and tested in the survey.	Involve theoretical analysis and experimental testing of more chords and rhythms.
Only about 35 responds to the survey.	Ask for more responds for the survey.

In reality, the music field does not strictly stick to a single scale. While keyboard instruments with strict notes like piano uses the Equal temperament, instruments of strings apply different scales in different scenarios. (Maltz, 2020) For example, due to the great harmony of chords produced by Just intonation, string players often choose Just intonation as the scale when playing chords. Overall, with the flexibility of the scale (given by perfect transposition), the

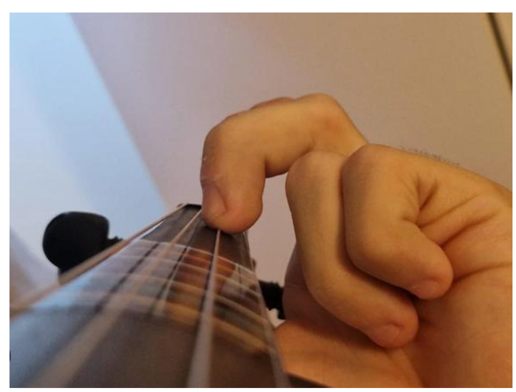

Figure 26 Pressing of a finger on a string to produce notes (r/violinist, 2018)

Equal temperament is the best fit scale for an instrument to choose, especially if the instrument could only be fixed to perform in one particular scale (e.g piano).

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