ODE and Time Solution

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1.0 Learning Outcomes

After studying this ACE-Lab learning guide, you should be able to demonstrate the Learning Outcomes listed in Table 1. On completion of the ACE-Lab learning guide, revisit Table 1 and confirm whether each Learning Outcome has been achieved.

Table 1: *Learning Outcomes from the ODE and Time Solution Learning Guide*

ID	Learning Outcome	Achieved
1.	Explain the purpose of ordinary differential equations (ODEs) in representing the dynamic behaviour of engineering systems.	☐
2.	Identify and interpret the key components of first- and second-order ODEs, including system inputs, outputs, derivatives and coefficients.	☐
3.	Determine the time solution of a first-order ODE and apply an initial condition to obtain the response of a dynamic system.	☐
4.	Interpret and sketch a system response from its time solution by identifying its initial value, steady-state behaviour and dynamic characteristics.	☐

2.0 Introduction

In previous Learning Guides, a low-pass filter (LPF) was introduced as a practical engineering tool used to reduce unwanted high-frequency noise from sensor measurements. The key outcome was that a noisy measurement could be transformed into a smoother signal, allowing a control algorithm to operate

using information that more accurately represents the underlying physical system.

For implementation on a microcontroller, the LPF was expressed in difference-equation form:

$$y[k] = \alpha u[k] + (1 - \alpha) y[k - 1]$$

which is computationally efficient and well suited to embedded systems. However, an important question naturally arises:

Where can this type of dynamic behaviour originate from?

One useful way of answering this question is to consider the continuous-time behaviour of a physical system. Historically, low-pass filtering could be implemented using an analogue resistor-capacitor (RC) circuit. The behaviour of this circuit can be described mathematically by:

$$\tau \frac{dy(t)}{dt} + y(t) = u(t)$$

where $\tau = RC$, and is known as the time-constant.

This equation is an example of an ordinary differential equation (ODE). An ODE describes how a quantity changes with respect to another variable, which in control engineering is most commonly time. In this example, the ODE describes how the output of the RC circuit evolves with time in response to an applied input.

The LPF therefore provides a useful example of how continuous-time mathematics can be connected to dynamic system behaviour. An analogue RC circuit can be represented using an ODE, its behaviour can be analysed in continuous time, and the resulting model can, where required, be converted into a discrete-time representation suitable for implementation on a microcontroller.

Importantly, the modelling approach extends far beyond the low-pass filter. Ordinary differential equations are widely used to describe the dynamic behaviour of electrical, mechanical, thermal, fluid and electromechanical systems. They provide a mathematical means of describing how system outputs evolve with time in response to inputs and changes within the system.

This Learning Guide therefore focuses on two fundamental ideas: how an ODE represents dynamic behaviour, and how its time solution describes the resulting system response. These concepts provide the mathematical foundations required to understand how dynamic engineering systems can be represented, analysed and, subsequently, modelled for control-system design.

This Learning Guide, together with the subsequent guides on continuous-time and discrete-time modelling, develops these ideas progressively. Later Learning Guides will build upon these foundations to develop continuous-time models and convert them into forms suitable for digital implementation.

The mathematics introduced in this guide may not initially appear intuitive. However, as the concepts are applied to practical engineering systems in subsequent Learning Guides, their meaning and usefulness will become increasingly apparent. With continued application, the mathematical processes introduced here become part of a structured engineering approach to modelling, analysis and control-system design.

3.0 Ordinary Differential Equations (ODEs)

Ordinary Differential Equations (ODEs) are mathematical equations that describe the relationship between a function and its derivatives with respect to one independent variable, usually time. They are used to model and analyse the dynamic behaviour of systems across a wide range of fields, including:

- Electrical circuits
- Mechanical motion
- Thermal systems
- Fluid flow
- Chemical processes

In each case, where time is the independent variable, the governing equations describe how a system response changes with time. For this reason, ODEs form one of the foundational mathematical tools of control engineering. They provide the language used to model system dynamics, predict behaviour, and design engineering solutions.

Why use ODEs for Control Engineering?

ODEs are fundamental to control engineering because they provide the mathematical framework used to model dynamic systems. Many physical processes, including mechanical, electrical, and thermal systems can be represented using first- and second-order ODEs, which describe how system variables evolve over time. First-order ODEs typically model simple processes such as RC circuits or thermal dynamics, while second-order ODEs capture systems with inertia, such as mass–spring–damper mechanisms or electric motors. The order of an ODE is defined by the highest derivative present in the equation. Solutions may be obtained analytically, producing an exact mathematical expression, or numerically, where methods such as Euler or Runge–Kutta approximate the solution at discrete points. In control engineering, these ODE models form the basis for representing and analysing both individual subsystems and complete control systems.

Once an ODE model has been developed, its solution can be used to investigate how the system responds to an input. This provides the basis for analysing dynamic behaviour and, in later stages, designing and tuning controllers to achieve specified performance requirements.

Mathematical Representation

When designing a closed-loop control system using modelling and simulation, use is made extensively of 1st and 2nd order linear-time invariant (LTI) ODEs. An ODE is used to describe the phenomenon of the rate of change of a physical quantity with respect to time.

Having a mathematical model of a system is typically the first step to designing a control system, and many practical engineering systems are modelled using ODEs. This mathematical model is usually developed based on a 1st principles approach, resulting in an ODE. Alternatively, a data-based approach using parameter estimation and system identification methods may be used (the latter is covered in the later Chapters). In engineering applications, when referring to the system processes, ODEs can be used to model almost all domains, e.g., thermal, fluids, solid mechanics and electrical circuits/devices. Additionally, ODEs can be used to model an actuator, e.g., a DC motor. In the later stages of designing a control system, ODEs are also used for other components, e.g., proportional integral derivative (PID) control and low pass filter (RC circuit) devices to filter noisy measurement data. Some examples of such ODEs are given in Figure 1.

The order of an ODE is determined by the highest derivative of the output that appears in the equation. A first-order ODE therefore contains a first derivative, whereas a second-order ODE contains a second derivative.

A general 1st order ODE is given by:

$$\frac{dy(t)}{dt} + \alpha y(t) = \gamma u(t) \tag{1}$$

A general 2nd order ODE is given by:

$$\frac{d^2y(t)}{dt^2} + \alpha\frac{dy(t)}{dt} + \beta y(t) = \gamma u(t) \tag{2}$$

where α, β and γ are known as the coefficients, with α, and α, β relating to the dynamics of the 1st and 2nd order ODEs, respectively, and with (α and γ), and (β and γ) relating to the system gains of the 1st and 2nd order ODEs, respectively, $y(t)$ is the measured system output, $u(t)$ is the known input and both are functions of continuous-time, denoted t, hence the notation (t).

In standard notation the system gain is denoted K. For 1st and 2nd order ODEs the process for determining the gain K will be considered in later chapters.

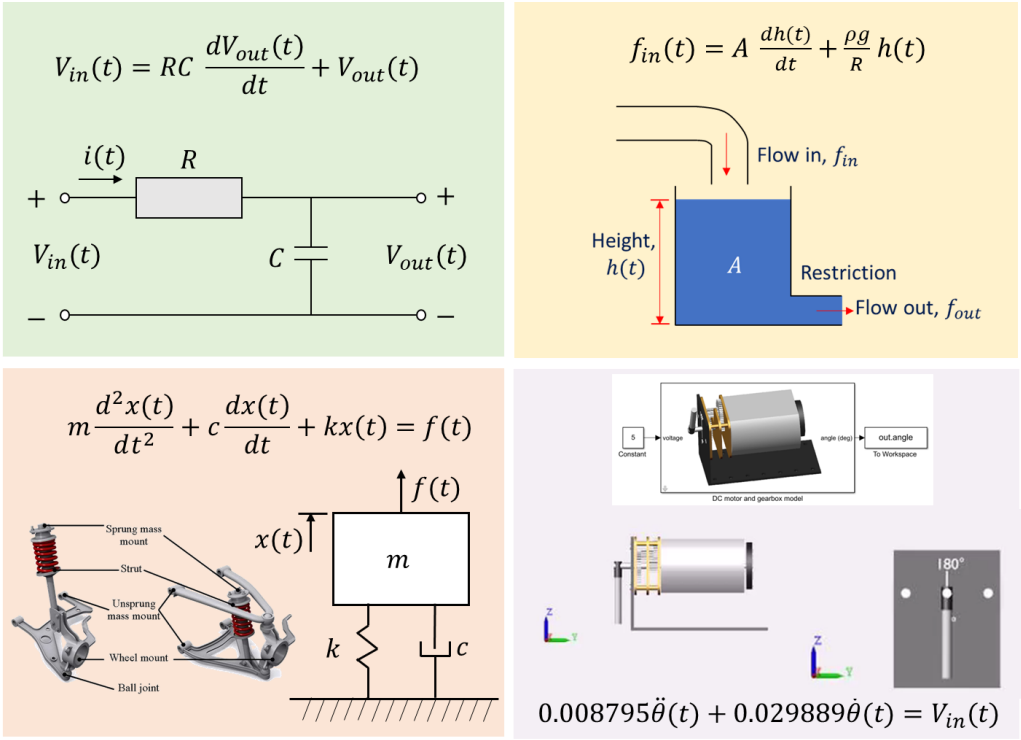


Figure 1: *Linear Time-Invariant (LTI) Ordinary Differential Equations (ODEs) are used to Mathematically Model Various Components of a Closed-Loop Control System, e.g., Low Pass Filter (RC circuit) (Top-Left), Water Tank Process (Top-Right), Suspension System Process (Bottom-Left) and DC Motor (Bottom-Right)*

Applied ODEs

Figure 1 presents several examples of engineering systems that can be represented using ODEs, including an RC circuit, water tank, mass-spring-damper system and DC motor.

Although these systems operate in different physical domains, the underlying purpose of the ODE is the same: to describe how a system variable changes with time in response to an input and the physical characteristics of the system.

For the **RC circuit**, the ODE describes how the capacitor voltage changes with time following an applied electrical input. The resistance and capacitance determine how quickly the voltage responds, meaning that the mathematical model captures the dynamic behaviour of the circuit.

For the **water tank**, the ODE describes how the liquid level changes with time as water enters or leaves the tank. Parameters such as the tank cross-sectional area and the relationship between flow rate and liquid level influence the resulting dynamic response.

For the **mass-spring-damper system**, the ODE describes how the position of the mass changes with time in response to an applied force. The mass, spring

stiffness and damping determine how quickly the system responds and whether its motion contains oscillatory behaviour.

For the **DC motor**, the ODE describes how a quantity such as rotational speed changes with time in response to an electrical input. Depending on the level of detail required, the mathematical model may represent only the dominant motor dynamics or may include both its electrical and mechanical behaviour.

These examples demonstrate that ODEs provide a common mathematical language for describing dynamic behaviour across different engineering systems. The physical quantities and parameters may change from one application to another, but the modelling principle remains the same: the ODE relates the system input, its physical characteristics and the resulting change in the system output with time.

Mathematical models can also be developed with different levels of fidelity. A relatively simple model may capture only the dominant behaviour required for a particular engineering task, whereas a higher-fidelity model may include additional physical effects and parameters. The appropriate level of modelling detail therefore depends on the purpose of the model and the accuracy required.

4.0 Time-solution

The time-solution of an ODE describes how a dependent variable changes with time. In control engineering, this variable may represent a physical quantity such as position, speed, temperature, voltage or liquid level. The time-solution therefore provides a direct link between the mathematical model of a system and the dynamic response that would be observed in practice.

An ODE itself describes the relationship between a system variable and its rate of change. Solving the ODE converts this relationship into an explicit expression for the variable as a function of time, such as $y(t)$. Where an initial condition is known, it can be applied to determine the specific response of the system.

There are several methods available for solving ODEs. In this Learning Guide, an analytical method is demonstrated for a simple first-order ODE. The purpose of the example is not primarily to practise algebraic manipulation or integration, but to show how an ODE can be transformed into a time-domain expression that describes the behaviour of a dynamic system.

The individual mathematical steps should therefore be viewed as a structured method for moving from:

ODE → general solution → initial condition → time solution → system response

Later Learning Guides will build on this relationship when mathematical models are used for simulation, analysis and control-system design.

Example 1: Time-solution of an ODE
Consider the following ODE, With the initial condition $y(0) = 1$.

$$\frac{dy}{dt} + 4y = 10$$

Determine the time-solution.

Step 1: Solve the homogeneous equation.
The homogeneous ODE is given by:

$$\frac{dy}{dt} + 4y = 0$$

This is a separable differential equation; therefore, this can be rewritten as:

$$\frac{dy}{y} - 4dt$$

Integrating both sides gives:

$$ln|y| = -4t + C_1$$

Taking the exponential on both sides to remove the logarithm gives the following:

$$|y| = e^{-4t+C_1} = e^{C_1}e^{-4t}$$

Let $C = e^{C_1}$, so the general solution to the homogeneous equation is:

$$y_h(t) = Ce^{-4t}$$

where C is an arbitrary constant.

Step 2: Solve for the particular solution

Now, solve for a particular solution to the non-homogeneous equation, i.e., $\frac{dy}{dt} + 4y = 10$. Considering a constant solution $y_p(t) = A$ and substituting $y_p(t) = A$ into the ODE gives the following:

$$0 + 4A = 10$$

Solving for A gives the following:

$$A = \frac{10}{4} = 2.5$$

Therefore, the particular solution is given by:

$$y_p(t) = 2.5$$

Step 3: General Solution

The general solution is the sum of the homogeneous solution and the particular solution, with this given by:

$$y(t) = y_h(t) + y_p(t) = Ce^{-4t} + 2.5$$

This is the general solution to the ODE.

Step 4: Applying the initial condition
As the initial condition is given by, $y(0) = 1$, this is substituted at $t = 0$ into the general solution:

$$1 = Ce^{-4(0)} + 2.5$$

Since $e^0 = 1$, this simplifies to:

$$1 = C + 2.5$$

Solving for C gives the following:

$$C = 1 - 2.5 = -1.5$$

Step 5: Final Time Solution
Substituting $C = -1.5$ back into the general solution, i.e.,

$$y(t) = -1.5e^{-4t} + 2.5$$

Final Solution:
The solution to the ODE $\frac{dy}{dt} + 4y = 10$ with the initial condition $y(0) = 1$ is given by:

$$y(t) = 2.5 - 1.5e^{-4t}$$

This is the final time solution that satisfies both the differential equation and the specified initial condition.

Time-Solution Response Sketch

When sketching the dynamic response of a time-solution, it is essential to consider several key factors to accurately capture the system's behaviour:

1. Determine the initial value: The initial condition provides the starting point of the system at $t = 0$. This is crucial for setting the initial value of the graph, as it represents the system's state before any changes occur. Determining $y(0)$ helps define the initial point on the plot and influences how the response evolves over time.

2. Determine the steady-state value: The long-term behaviour of the system, also known as the steady-state, shows how the system behaves as time approaches infinity. Evaluating $lim_{t\to\infty} y(t)$ reveals whether the

system stabilizes, decays to zero, or grows indefinitely. This information is essential for understanding the system's final value and where the curve will settle over time.

3. Identify decay, growth or oscillation: If the time-solution contains exponential terms (such as e^{-kt}), it indicates a decaying or growing response depending on the sign of the exponent. For a negative exponent (e.g., e^{-t}), the response will decay over time, meaning it starts from an initial condition and asymptotically approaches a final value. If the exponent is positive (e.g., e^{t}), the response exhibits exponential growth, increasing rapidly as time progresses. The rate of decay or growth determines the steepness of the curve.

4. Calculate intermediate points if required: If the solution contains sinusoidal terms (such as $sin(t)$ or $cos(t)$, the system exhibits oscillations. In this case, the graph alternates between maxima and minima, with a consistent wavelength or possibly a decaying amplitude. Understanding whether oscillations persist or decay over time is important for accurately depicting the wave-like behaviour in the response. (Note that the details to sketch an oscillatory response are not provided in this Chapter.)

5. Sketch and label the response: Clear labelling of the graph is essential for interpretation. The axes should be properly labelled, typically, the horizontal axis represents time t (independent variable) and the vertical axis represents the response $y(t)$ (dependent variable). If the problem involves physical quantities, units should be indicated, such as seconds for time or meters for distance. Additionally, labelling the curve itself can help differentiate multiple responses if comparing different solutions.

When sketching, use key points such as the initial condition, steady-state value, and any intermediate points to guide the shape of the curve. The overall sketch should reflect the system's behaviour, such as whether it decays, grows, or oscillates. The curve should be drawn smoothly and should follow the expected dynamic trends of the system.

Example 2: Sketch system response of a time-solution
Sketch the system response of the following time-solution:

$$y(t) = 2.5 - 1.5e^{-4t}$$

Step 1: Find $\boldsymbol{y(0)}$: The initial condition gives the starting point of the graph.

$$y(0) = 2.5 - 1.5e^{0} = 2.5 - 1.5 = 1.0$$

So, at $t = 0$, the response starts at $y(0) = 1.0$

Step 2: Find $\boldsymbol{y(\infty)}$: Evaluate $lim_{t\to\infty}$, as t approaches infinity, the exponential term e^{-4t} decays to zero. Therefore, the system response stabilises at:

$$\lim_{t\to\infty} y(t) = 2.5$$

This indicates the system's final steady-state value.

Step 3: Identify decay, growth and oscillation: There are no sinusoidal terms in the given solution, which means that the system does not exhibit oscillatory behaviour. Therefore, the response is smooth without alternating between peaks and troughs.

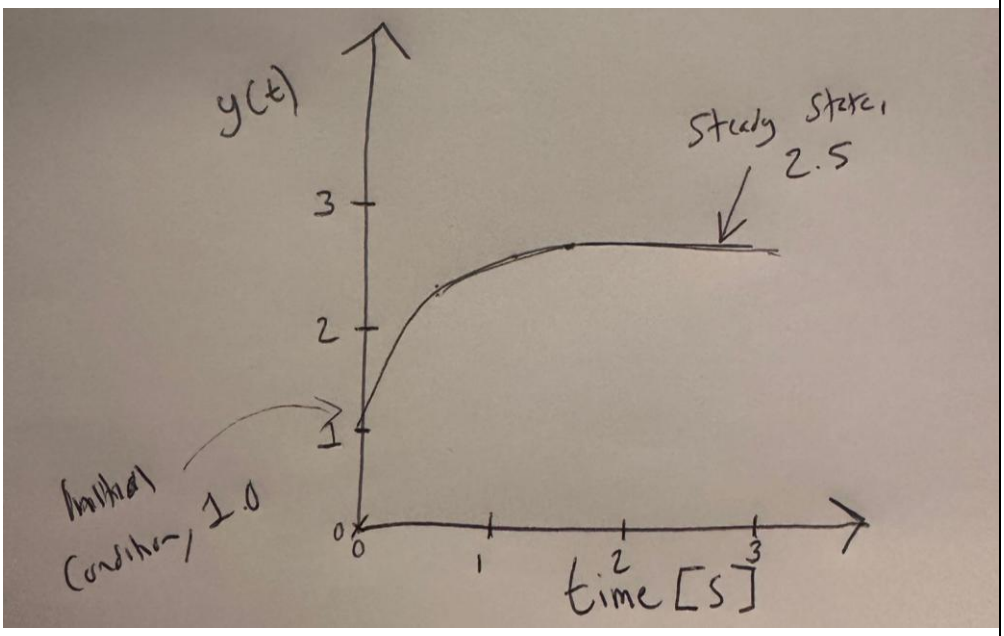
Step 4: Calculate a few intermediate points if required: Data points for the system response can be determined using $y(t) = 2.5 - 1.5e^{-4t}$ to allow for an approximate sketch, e.g.,

$$y(t) = 2.5 - 1.5e^{-4(0.25)} = 1.95$$

The table below has significant data points determined to approximate the system response.

t	0.0	0.25	0.50	0.75	1.00	1.25	1.50
$y(t)$	0	1.95	2.30	2.43	2.47	2.49	2.50

Step 5: Sketch and label the response:



MATLAB Script for Time-Solution Response

The time response obtained analytically in Example 2 can also be verified numerically using MATLAB. The ode45 solver numerically integrates the ODE over the specified time interval, allowing the calculated response to be compared with the hand-derived time solution and sketch.

i. Plot the graphical output (system response) of the time-solution

MATLAB Script: the following script is used to determine the system response and to plot the graphical output

```
% Define the ODE as a function handle
% dy/dt = 10 - 4*y
ode = @(t, y) 10 - 4*y;

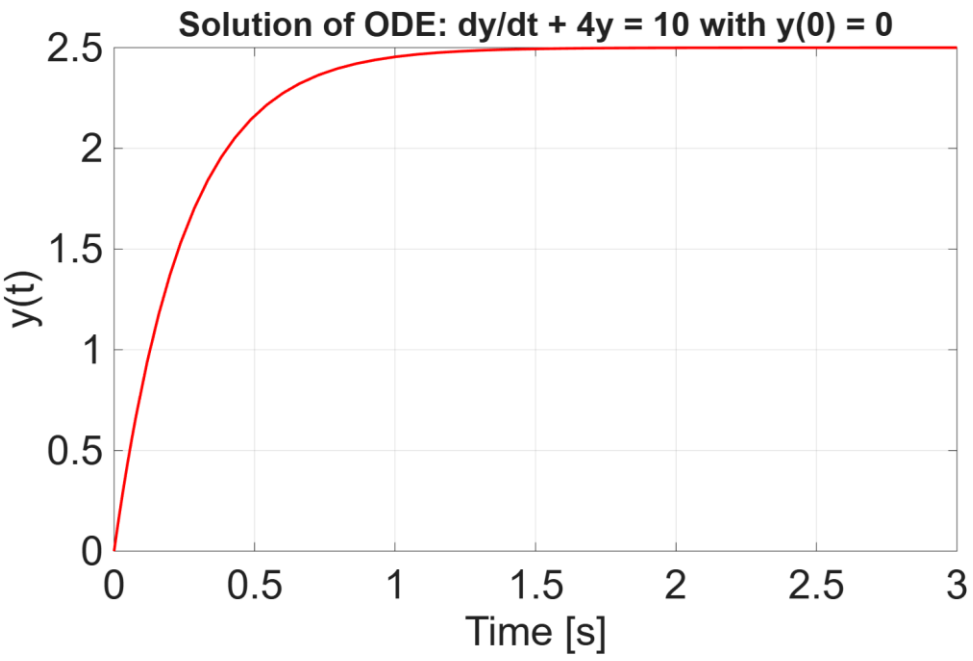
% Set the time span for the solution
tspan = [0 3];  % Solve from t = 0 to t = 3

% Set the initial condition y(0) = 0
y0 = 0;

% Use ode45 to solve the ODE
[t, y] = ode45(ode, tspan, y0)

% Plot the solution, change the font size of the labels and
title
figure;
plot(t, y, 'r-', 'LineWidth', 2);
set(gca, 'FontSize', 30); % Change font size of axis tick
labels
xlabel('Time [s]', 'FontSize', 30); % Change font size of X-
axis label
ylabel('y(t)', 'FontSize', 30); % Change font size of Y-axis
label
title('Solution of ODE: dy/dt + 4y = 10 with y(0) = 0',
'FontSize', 25);
grid on;
```

Solution:



5.0 Exercises

Consider the following time-solutions:

a) $y(t) = 1.5 - 1.5e^{-2t}$

b) $y(t) = 1.5 - 0.5e^{-t}$

c) $y(t) = 2.5 - 1.5e^{4t}$

For each of the following time solutions, undertake the following tasks:

a) determine $y(0)$
b) determine the steady-state value $y(\infty)$
c) identify whether the response contains exponential growth, decay or oscillation
d) calculate several intermediate values
e) sketch the response
f) verify the response using MATLAB

6.0 Summary

This Learning Guide introduced ordinary differential equations (ODEs) as a mathematical means of describing the dynamic behaviour of engineering systems. First- and second-order ODEs were introduced, together with examples showing how they can be used to represent systems from different engineering domains.

The relationship between an ODE and its time solution was then considered. By solving an ODE and applying an appropriate initial condition, an expression can be obtained that describes how the system output changes with time. The time solution therefore provides an important link between the mathematical representation of a system and the dynamic response that would be observed in practice.

The interpretation of time solutions was also introduced. Initial conditions, steady-state behaviour, exponential growth or decay, and oscillatory behaviour can all provide useful information when predicting and sketching a system response. MATLAB can subsequently be used to verify these responses numerically and graphically.

These concepts provide the mathematical foundations required for later Learning Guides on continuous-time and discrete-time modelling. The key progression introduced in this guide can be summarised as:

ODE → time solution → system response

Understanding this relationship is an important step towards using mathematical models to analyse, simulate and ultimately design control systems.

Recommended Citation

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