Advanced Microeconomics Review

Tu Ni

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1 Strategic Game

1.1 Classic Strategic Games

- Nash equilibrium
  - Typical examples: Battle of Sex, Prison Dilemma, Hawk-Dove, Matching Pennies
  - Assumption: belief about typical component's action is correct
  - Interpretation: if everyone else adheres to it, no individual wishes to deviate from it
  - Characterization: argue for some profiles, no player will not deviate, and for other profiles, some player will deviate
  - Existence: action set is nonempty, compact and convex; utility function is continuous and quasi-concave on actions
- Strictly Competitive Game
  - Maxminimizer of player: optimize the worst-case payoff in two player zero-sum game
  - Property: a^* is a NE if and only if for both players, a_i^* is his maxminimizer
- Bayesian Game
  - Typical Examples: auction(each player's evaluation is the type)
  - Assumption: player has a prior belief on the state ω, then he receives the signal t and update his posterior belief; for applications, player holds the preference relation over lotteries, which takes the form of expected utility function
  - Equilibrium: s^* is a NE if and only if for any player i, conditioning on any possible type t_i, given other players' strategies, $s_i^*(t_i)$ is the best response compared to all strategies over posterior distribution of states.
- Mixed Strategy Nash Equilibrium
  - Typical Examples: matching pennies
  - Definition: a mixed NE is a NE of its mixed extension
  - Assumption: each player can assign a probability distribution over his action sets, and then the utility is calculated as expected value over all possible action profile(players' distributions are independent)
  - Theorem: every finite strategic game has a mixed NE

  - Equilibrium: α^* is a mixed NE if and only if for any player i, given other players' strategies, α_i^* is the best response compared to all pure strategy
  - Property: every action in the support of any player's mixed NE strategy yields that player the same equilibrium payoff
  - Characterization: calculate the best-response equations for all the players, and then take the intersection

- Correlated Equilibrium
  - Assumption: compared with mixed strategy, it does not require independence on players' strategy marginal distribution. Instead, it's characterized by joint distribution such that players could somehow avoid bad outcome
  - Definition: there is a set of states Ω and π is a probability measure on Ω, each player has an information partition P_i over Ω such that his strategy σ_i will perform the same action for states ω lie in the same cell of P_i
  - Equilibrium: σ is a CE if and only if for any player i, conditioning on any information cell P_i with $\pi(P_i) > 0$, σ_i is the best response compared to other actions over posterior distribution of states.
  - Property: any convex combination of CE payoff profiles is a CE payoff profile

1.2 Rationalizability

- Rationalizability
  - Definition: a product subset R of action set A is rationalizable if for any player i, any action of his rational set R_i, there exists a corresponding belief about other players, such that a_i is a best response compared to other actions
  - Interpretation: the notion of rational determines not what actions should actually be taken, but what actions can be ruled out with confidence

- Never-Best Response
  - Definition: given X, some actions a_j can be iteratively eliminated if for each $\mu \in \Delta X_{-j}$, there is $a_j^* \in A_j$ gives higher expected utility than a_j
  - Interpretation: compared with rationalizability, IENBR generates the final set R^* in the other direction by iteratively removing bad actions, and clearly NE will not be eliminated
  - Property: in dominance-solvable games, NE $= R^*$

- Strictly Dominated Action
  - Definition: action a_i of player i is strictly dominated if there is a mixed strategy α_i such that for any other players' actions, α_i gives higher expected utility, and we can also construct the final rational set R^* by iteratively removing strictly dominated actions
  - Property: SDA = NBR

- Summary: we can define rationalizability as the consequence of common knowledge of rationality, or as the result of iterative elimination procedure. Hence, we have $R^* =$ IENBR $=$ IESDA

2 Interactive Epistemology

2.1 Knowledge Model

- Partitional Information Structure
  - P1: $\omega \in P(\omega)$ for every $\omega \in \Omega$
  - P2: if $\omega' \in P(\omega)$ then $P(\omega') = P(\omega)$
- Knowledge Operator
  - Definition: $KE = \{\omega \in \Omega \mid P(\omega) \subseteq E\}$
  - Interpretation: for an event E, player knows E at ω if $P(\omega) \in E$, so we care about when the player knows E, which characterized by KE
  - Property
    * K1: $K\Omega = \Omega$
    * K2: if $E \subseteq F$, then $KE \subseteq KF$
    * K3: $KE \cap KF = K(E \cap F)$
    * K4: $KE \subseteq E$
    * K5: $KE \subseteq KKE$
    * K6: $\neg KE \subseteq K \neg KE$
- Common Knowledge
  - Definition: we denote $K_i E$ as i knows E, and $\cap_{i \in N} K_i E$ as the event that E is mutually known, then we have representation of common knowledge

$$CKE = KE \cap KKE \cap KKKE \cap ....$$

  - Alternative Definition: an event $F \in \Omega$ is self-evident if $KF = F$, then under K4, an event E is CKE at ω if it includes a self-evident event F containing ω
  - Property: E is self-evident if and only if E is a union of members of the partition induced by P_i for all player i
  - Agreeing to Disagree: $CKE^{[\mu_i;\mu_j]} = \emptyset$ if $\mu_i \neq \mu_j$

2.2 Epistemic Foundation

- Model of Knowledge
  - Definition: given a strategic game, we need some parameters Ω is the set of states, P_i is player i's information partition, $a_i(\omega)$ is i's action at ω, $\mu_i(\omega)$ is i's belief at ω
  - Interpretation: each $\omega \in \Omega$ consists of the description of each player's knowledge, action and belief
  - Rationality: we say i is rational at ω if $a_i(\omega)$ is a best response of player i to $\mu_i(\omega)$ in $\Delta[a_{-i}(P_i(\omega))]$, in other words, i's action at the state maximizes his expected utility with respect to the belief that i holds at that state, where the belief is required to be consistent with his knowledge
  - Rational Set: we define $R_i = \{\omega : \textit{ i is rational at } \omega\}$
- Epistemic Conditions for Nash Equilibrium

- Definition: let $\omega \in \cap_{i\in N}\{R_i \cap K_i[a_{-i} = a_{-i}(\omega)]\}$, then $a(\omega)$ is a NE
- Interpretation: if each player is rational and knows the action choices of the others, then the players' choices constitute a NE

• Epistemic Conditions for Mixed Strategy Nash Equilibrium

- Definition: suppose each player's belief is consistent with his knowledge, and let $\omega \in \cap_{i,j=1,2;i\neq j}\{K_j R_i \cap K_j[\mu_i = \mu_i(\omega)]\}$, then $\mu(\omega)$ is a mixed-strategy NE
- Interpretation: if the rationality of the players and their 'consistent' conjectures are mutual knowledge, then the conjectures consitute a mixed-strategy NE

• Epistemic Conditions for Rationalizability

- Definition: let $\omega \in CKR$, then $a(\omega)$ is a rationalizable strategy profile
- Interpretation: if a state is common knowledge of rationality, then we can find a self-evident set to construct rationalizable subset Z

3 Extensive Games

3.1 Extensive Games with Perfect Information

• Model

- History: set of sequences that each member is a history h, whose component is an action
- Strategy: a complete plan s that specifies the action chosen by the player in every contingency, even for histories that never reached

• Nash Equilibrium

- Equilibrium: for every strategy s_i^* of player i, s_i^* is the best response to s_{-i}^*
- Property: s^* is NE if and only if it is a NE of strategic game derived from extensive game

• Subgame Perfect Equilibrium

- Definition: for every subgame with $P(h) = i$, we have s_i^* is the best response to s_{-i}^* for player i, then s^* is SPE
- Equilibrium: s^* is SPE of a finite-horizon game if and only if $\forall h \in H$ with $P(h) = i$, s_i^* is the best response to s_{-i}^*, compared with other strategies that differs from s_i^* only in the first action of subgame
- Property: s^* is a SPE if and only if it's a NE in every subgame

3.2 Extensive Games with Imperfect Information

• Model

- Interpretation: players only have partial information about the history, when choosing an action, he need to form an expectation about the unknowns, not only derived solely from the players' equilibrium behavior in the future, but also from past behavior inconsistent with equilibrium
- Information Partition: each player has his own information structure $\mathcal{I}_i$ such that he cannot distinguish the histories in the same cell with $A(h) = A(h')$

- Perfect Recall: at every point every player remembers whatever he knew in the past, which is a common assumption in imperfect game
- Strategy
  - Pure Strategy: compared with perfect information, now for every contingency, the strategy is a function from every cell in the information partition to its corresponding action
  - Mixed Strategy: there is an outcome-equivalent behavioral strategy with Kuhn's theorem, and vice versa

- Perfect Bayesian Equilibrium
  - Interpretation: assume all the information partition cells are singleton, and the only uncertainty comes from initial chance move, then the problem can discussed in Bayesian setting
  - Equilibrium: a PBE is the combination of a strategy profile s and a belief assessment profile μ, such that for every player i, he satisfies sequential rationality (at each of his information sets, s_i is a best response to s_{-i}, given his belief μ_i at that information set) and belief consistency (at information sets on the equilibrium path, his belief μ_i is derived from Bayes' rule using the strategy profile s; at information sets off the equilibrium path, his belief μ_i is derived from Bayes' rule using the strategy profile s where possible)

- Sequential Equilibrium
  - Equilibrium: a SE is an assessment (β, μ) with a trembling sequence of totally mixed behavioral strategies $\beta^k \longrightarrow \beta$ such that for every player i, for every his information set I_i, he satisfies sequential rationality (β_i is his best response to β_{-i}, given that belief $\mu(I_i)$) and sequential consistency ($\mu^k(I_i) \longrightarrow \mu(I_i)$ where $\mu^k(I_i)$ is the belief assessment which derived from β^k using Bayes' rule)
  - Property: the notion of SE requires that the beliefs at different information sets not reached in the equilibrium be derived from the same trembling sequence while the notion of PBE imposes no such restriction

- Perfect Equilibrium
  - Equilibrium: compared with SE, it's required that for every player i, for every information set, $\beta_i(I_i)$ is best response to not only β_{-i}, but also β^k_{-i}
  - Property: $PE \subseteq SE \subseteq PBE \subseteq SPE \subseteq NE$

4 Implementation Theory

- Model
  - Environment
    - N: set of players
    - C: set of outcomes
    - $\mathcal{P}$: set of preference profiles or states
    - $\mathcal{G}$: set of games with consequences in C
  - Choice Rule: $f : \mathcal{P} \longrightarrow C$ is a function that assigns a subset of C to each preference profile in $\mathcal{P}$

- S-implementation

- S-implementation: the choice rule f is S-implementable in $\langle N, C, \mathcal{P}, \mathcal{G}\rangle$ if $\exists G \in \mathcal{G}$ with outcome function g such that
$$g(S(G, \succeq)) = f(\succeq), \forall \succeq \in \mathcal{P}$$
- Truthfully S-implementation: the choice rule f is truthfully S-implementable in $\langle N, C, \mathcal{P}, \mathcal{G}\rangle$ in which $\mathcal{G}$ is a set of strategic game forms for which the set of actions of a player i is $\mathcal{P}$, if $\exists G \in \mathcal{G}$ with outcome function g such that
$$g(a^*) \in f(\succeq) \cap g(S(G, \succeq)), \forall \succeq \in \mathcal{P}$$
where $a_i^* = \succeq$ for each $i \in N$
- Nash Implementation: if we restrict the solution concept to NE
- Revelation Principle: if f is Nash-implementable, then it's truthfully Nash-implementable.

- Maskin's Monotonicity
  - Definition: a choice rule $f : \mathcal{P} \longrightarrow C$ is monotonic if whenever $c \in f(\succeq)$ and $c \notin f(\succeq')$, there is some player $i \in N$ and some outcome $b \in C$ such that $c \succeq_i b$ but $b \succ_i' c$
  - Property: every Nash-implementable choice rule must be monotonic

5 Revealed Preference Analysis

- Afriat's Theorem
  - Idea: A utility function U is locally non-satiated if at every bundle x and for any open neighborhood N of x, there is a y such that $U(y) > U(x)$. Suppose the customer maximizes a locally non-satiated utility function. Then there will be some observable restrictions on the data and we'd like to know the necessary and sufficient condition to recover such utility function from data.
  - GARP: If $p^t x^t \geq p^t x^s$, then we say x^t is directly revealed preferred to x^s. Moreover, the data $\mathcal{O}$ is said to obey GARP if the only revealed preferred cycles are weak revealed preferred cycles.
  - Theorem: A data set $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ is consistent with the maximization of a locally non-satiated utility function(rationalizable) if and only if it obeys the generalized form of revealed preference(GARP).
    * Necessary: If $x^t \succeq x^s$, then $U(x^s) \leq U(x^t)$, similar argument for strictly preferred. Therefore whenever there is a RP cycle, we have a utility cycle, which means all the utilities must equal.
    * Sufficient: It's obvious that we can recover a preference from data, and we show such preference can be extended from $\mathcal{X}$ to R^l in a way that rationalizes the data. Specifically, we need two steps:
      1. The LP is feasible
      $$\phi^k \leq \phi^t + \lambda^t p^t (x^k - x^t), \lambda^t > 0, \forall k \neq t$$
      For simplicity, we consider the case of strict preference
      2. Given the decision variables of LP, we can construct a continuous, strictly increasing, piecewise-concave utility function that rationalizes the data:
      $$U(x) = \min_{(p^t, x^t)} \{\phi^t + \lambda^t p^t (x - x^t)\}$$

- Strongly Separable Utility
  - Idea: A utility function is additively separable if $U(x) = \sum_i v_i(x_i)$ where $v_i(x)$ is differentiably increasing and concave.

- Claim: The data $\mathcal{O}$ is rationalizable by an additively separable function if and only if there are numbers $\beta_i^t > 0$, for all (i,t), such that
    1. whenever $x_i^t > x_i^s$, then $\beta_i^t \leq \beta_i^s$ (Decreasing function)
    2. for all $t \in T$, we have $\frac{\beta_i^t}{p_i^t} = \frac{\beta_j^t}{p_j^t}$ (First order condition)

- Quasi-linear Utility Maximization
  - Idea: We move forward to require the utility function be quasi-linear, which is maximized by customer as
$$x^t \in argmax\{F(x) - px\} \; \forall t \in T$$
  - Law of Demand: for any subset of observations $\{(p^{t_i}, x^{t_i})\}_i^N = 1$ chose from data, the following inequality holds
$$p^{t_1}(x^{t_1} - x^{t_2}) + \cdots + p^{t_N}(x^{t_N} - x^{t_1}) \leq 0$$
  - Theorem: Given a finite set of observations, the following statements are equivalent:
    1. $\mathcal{O}$ admits a quasi-linear rationalization
    2. there exists a function F such that for any $x^t, x^s \in \mathcal{X}$, we have
$$F(x^t) - p^t x^t \geq F(x^s) - p^t x^s$$
    3. $\mathcal{O}$ obeys law of demand

    Proof. (3) implies (1): define F as the minimum of a finite set of strictly increasing affine functions.

6 Decision Making under Risk and Uncertainty

- Axiomatic Foundations
  - Independence Axiom: Given two lotteries π and φ, we denote by $a\pi + (1-a)\varphi$ the lottery where outcome $x \in X$ occurs with probability $a\pi(x) + (1-a)\varphi(x)$, and independence axiom says for $a \in (0,1)$
$$\pi \succ \rho \Longleftrightarrow a\pi + (1-a)\varphi \succ a\rho + (1-a)\varphi$$
  - Subjective Expected Utility: under independence axiom, there exists a utility function u such that
$$\pi \succeq \rho \Longleftrightarrow \sum_{x \in X} u(x)\pi(x) \geq \sum_{x \in X} u(x)\rho(x)$$
    The function u is unique up to affine transformations and this model characterizes the objective expected utility over risk.

    The proof of three state case has three steps:
    1. Suppose $e_1 \succ e_2 \succ e_3$, and we get $e_1 \succ (t, 0, 1-t) \succ e_3$
    2. If $t'' > t'$, then $(t'', 0, 1-t'') \succ (t', 0, 1-t')$, so indifference curves are parallel straight lines
    3. For any $(a,b,c) \sim (t, 0, 1-t)$, define $v(a,b,c) = t$, such that
$$v(a,b,c) = \theta v(te_1 + (1-t)e_3) + (1-\theta)v(se_1 + (1-s)e_2) = av(e_1) + bv(e_2) + cv(e_3)$$
  - Objective Expected Utility: This model characterizes the subjective expected utility in uncertain environment.
- Risk Aversion

- Definition: The agent is said to be risk averse if, given any lottery π, he always prefers the certain outcome $\bar{x} = \sum_{x \in X} \pi(x)x$ to the lottery π, which means

$$u(\bar{x}) \geq \sum_{x \in X} \pi(x)u(x)$$

This is true if and only if the utility function is concave.

- Example of Demand for Insurance: an agent has wealth W in the good state and w in the bad state, and he can buy insurance z with premium q. In particular, he has to choose z to maximize his expected utility

$$p_G u(W - qz) + p_B u(w - qz + z)$$

suppose the insurance is fair, so that expected profit is zero with

$$p_G(qz) + p_B(qz - z) = 0$$

if the agent is risk averse, then he should choose z so that

$$W - qz = w - qz + z$$

• Comparison of Utility Functions

- Idea: In order to compare different utility functions, we need to quantify the level of risk aversion, or the preference over certainty, so we introduce certainty equivalent of π as $C(\pi)$ such that $u(C(\pi)) = \sum \pi(x)u(x)$. In this way, we could define risk premium as the difference between expected value and certainty equivalent: $P(\pi) = \bar{x} - C(\pi)$. Moreover, the coefficient of risk aversion ρ_u is defined by $\rho_u(x) = -\frac{u''(x)}{u'(x)}$.
- Proposition: The following three criteria to compare utility functions are equivalent
  1. For all lotteries π, $P_u(\pi) \geq P_v(\pi)$
  2. $\rho_u(x) \geq \rho_v(x), \forall x$
  3. There exists an increasing concave function h such that $h \circ v = u$

• Stochastic Dominance

Instead of comparing different utility functions, from the perspective of consumers, we would like differentiate various of lotteries. We will present three dominance conditions from strong to weak.

- Monotone Likelihood Ratio Order
  * Definition: $\frac{\pi(x,\theta'')}{\pi(x,\theta')}$ is increasing in x whenever $\theta'' > \theta'$.
- First Order
  * Definition: if for all increasing utility functions u, we have

$$\int u(x)\pi(x)dx \geq \int u(x)\tilde{\pi}(x)dx$$

  we say π dominates $\tilde{\pi}$ by first order stochastic dominance.
  * Necessary and Sufficient Condition: the lottery π dominates $\tilde{\pi}$ by first order stochastic dominance if and only if $F_\pi(x) \leq F_{\tilde{\pi}}(x)$ for all $x \in [r_0, r^0]$, where $F_\pi(x) = \int_{r_0}^{x} \pi(t)dt$.
  * Interpretation: for any outcome x, the curve of cumulative distribution $F_\pi(x)$ is always below the other one, which means the consumer always has better chance to get higher utility.
- Second Order

* Definition: if for all increasing and concave utility functions u, we have

$$\int u(x)\pi(x)dx \geq \int u(x)\tilde{\pi}(x)dx$$

we say the lottery π dominates $\tilde{\pi}$ by second order stochastic dominance.
* Necessary and Sufficient Condition: the lottery π dominates $\tilde{\pi}$ by second order stochastic dominance if and only if $\int_{r_0}^{r} F_\pi(x) \leq \int_{r_0}^{r} F_{\tilde{\pi}}(x)$ for all $r \in [r_0, r^0]$. we say π dominates $\tilde{\pi}$ by second order stochastic dominance.
* Interpretation: similarly with first order dominance that for small x, lottery π should have lower chance but higher chance for large x, but the utility is concave, and thus not worthwhile to put too much weight on large x, so we allow the cumulative function F_π to catch up as a sense of trade-off; this dominance relation is not necessarily comparable because it is possible to have opposite inequalities for different x.

– Mean Preserving Spread
  * Idea: it's very common that two lotteries have the same(or almost similar) means, and we would like to explore some properties.
  * Definition: if compound lottery G has first stage outcome distribution F and second stage zero mean distribution H_x, we say G is a mean preserving spread of F.
  * Proposition: the following conditions are equivalent given two distributions F and G with the same mean
    1. G is a mean preserving spread of F
    2. for all concave functions u, we have

    $$\int_{r_0}^{r^0} u(x)dF(x) \geq \int_{r_0}^{r^0} u(x)dG(x)$$

    3. F second order stochastic dominates G
    4. For all $r \in [r_0, r^0]$, we have

    $$\int_{r_0}^{r^0} F_\pi(x)dx \leq \int_{r_0}^{r^0} G_\pi(x)dx$$

7 Monotone Comparative Statics

* One-dimensional Comparative Statics
  – Idea: we are interested in how $\arg\max_{x \in X} f(x; s)$ varies with some parameter s. For instance, a firm makes the decision to maximize its profit by $f(x; c) = xP(x) - cx$ or $f(p_1; p_2) = (p_1 - c_1)D_1(p_1; p_2)$.
  – Definitions
    * Set Comparison: S'' donimates S' in the strong set order if for any $x'' \in S''$ and $x' \in S'$, we have $\max\{x'', x'\} \in S''$ and $\min\{x'', x'\} \in S'$.
    * Single Crossing Property: ϕ has the single crossing property if $\phi(s') \geq (>)0 \Longrightarrow \phi(s'') \geq (>)0$ where $s'' > s'$.
    * Single Crossing Differences: the family of functions $\{f(, s)\}_{s \in S}$ obeys single crossing differences if for all $x'' > x'$, the function $\delta(s) = f(x''; s) - f(x'; s)$ is a single crossing function. Strictly increasing function preserves single crossing differences.

- * Increasing Differences: the family of functions $\{f(,s)\}_{s\in S}$ obeys increasing differences if for all $x'' > x'$, the function $\delta(s) = f(x''; s) - f(x'; s)$ is an increasing function.
- – Theorem: the family of functions $\{f(,s)\}_{s\in S}$ obey single crossing differences if and only if $\arg\max_{x\in Y} f(x; s)$ is increasing in s for all $Y \subseteq X$.

  Proof. For necessity, just choose $Y = \{x', x''\}$, then the result follows from the definition; For sufficiency, we denote $O'' = \arg\max_{x\in Y} f(x; s'')$ and $O' = \arg\max_{x\in Y} f(x; s')$. Assume $s'' > s'$ and $x'' \in O'', x' \in O'$. We have to show $\max\{x'', x'\} \in O''$ and $\min\{x'', x'\} \in O'$, so we only need to consider $x' > x''$. Since $x' \in O'$, we have $f(x'; s') \geq f(x''; s')$. By single crossing differences, $f(x'; s'') \geq f(x''; s'')$ leads to $x' \in O''$. With the similar argument we can show $x'' \in O'$. □

  Applications

  * $f(x; c) = xP(x) - cx$: it's obvious that $f(, -c)$ obeys increasing differences and thus decreasing in c. Therefore, as marginal cost increases, the profit-maximizing output will decrease.
  * $f(p_1; p_2) = (p_1 - c_1)D_1(p_1; p_2)$: if $\ln f(, p_2)$ obeys increasing differences and then increasing in p_2. Therefore, as other firm increases the price, the firm's optimal price will also increase.

- Optimization under Uncertainty
  - Idea: suppose the family of functions $\{v(,s)\}_{s\in S}$ obeys increasing or single-crossing differences, so the optimal decision x is also increasing in s. In practice, we usually don't know s in advance, so x is chosen to maximize some expected value as

  $$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)ds$$

  where θ controls the density distribution of s. Intuitively, if higher states are more likely with respect to $\lambda(s, \theta)$, then the optimal decision should also be higher.
  - Monotonicity in Density Function
    * Definition: a family of density functions $\{\lambda(, \theta)\}_{\theta\in\Theta}$ is said to be ordered by FOSD if $\lambda(s, \theta'')$ first order stochastically dominates $\lambda(s, \theta')$ whenever $\theta'' > \theta'$. Similarly, we can also define the MLR order correspondingly. In particular, we would like to know the condition that leads to the monotonicity of optimal decision with respect to density function.
    * Theorem 1: suppose the family of value functions obeys increasing differences and the family of density functions is ordered by FOSD, then the family of expected value functions also obeys increasing differences, and consequently, the optimal decision is increasing in θ.
    * Theorem 2: suppose the family of value functions obeys single crossing differences and the family of density functions is ordered by MLR, then the family of expected value functions obeys single crossing differences, and consequently, the optimal decision is increasing in θ.
  - Applications
    * $f(x; c) = xP(x) - cx$: consider $v(x; -c) = u(f(x; -c))$, even though $f(x; -c)$ obeys increasing differences, given u as a increasing utility function, we can only conclude $v(x; -c)$ obeys single crossing differences. If $\lambda(c, \theta)$ performs in the MLR sense, then the firm will choose to produce less when higher c becomes more likely.
    * $f(x; s) = (w - x)r + xs$: an investor with wealth w is given r as the payoff of safe asset and s as the payoff of the risky asset, which is uncertain. If $\lambda(s, \theta)$ performs in the MLR sense, then the investor will choose to put more in risky asset when higher s becomes more likely.
- Multi-dimensional Monotone Comparative Statics

- Definition: when the decision output is a vector, the supremum and infimum of two decisions are derived as element-wise maximum and minimum respectively. A function F is said to be supermodular if $F(x' \vee x'') - F(x'') \geq F(x') - F(x' \wedge x'')$. The necessary and sufficient condition for supermodular is nonnegative cross derivatives.
- Idea: consider a production problem with $f(x; w) = pF(x) - wx$ where p and w are output and input prices respectively. x is the input of many resources, and $F(x)$ is the output of product quantity. We care about the condition of F that leads to monotonicity of x with respect to w.
- Theorem: suppose F is supermodular, then if $w' < w''$, we have $\arg\max f(x; w') \geq \arg\max f(x; w'')$ in the strong set order.

* Increasing Maps
  - Tarski Fixed-Point Theorem: suppose ϕ is increasing function from X to X, then the set of fixed points is non-empty. In particular, $\overline{x} = \sup\{x \in X : x \leq \phi(x)\}$ is the largest fixed point while $\underline{x} = \inf\{x \in X : x \geq \phi(x)\}$ is the smallest fixed point.
  - Increasing Fixed-Point Theorem: suppose $\phi(, t)$ is increasing in (x, t), then the largest and smallest fixed points of $\phi(, t)$ are both increasing in t.

    Remark. ϕ only need be increasing but not necessarily continuous and X need not be convex.

* Games of Strategic Complements
  - Definition: G is a game of strategic complements if for every player a, there is an increasing function $\phi_a : X_{-a} \longrightarrow X_a$, such that $\phi_a(x_{-a}) \in BR_a(x_{-a})$ for all x_{-a}.
  - Equilibrium: every game of strategic complements has a pure strategy Nash equilibrium. We can construct $\phi(x) = (\phi_a(x_{-a}))_{a \in A}$ as an increasing function and then apply Tarski theorem.
  - Condition for Strategic Complements: suppose that for every player a, his utility function u_a is continuous in x_a and has single crossing differences in $(x_a; x_{-a})$, then it's a game of strategic complements and there are the largest and smallest NE decisions.

    Proof. By the theorem of monotone comparative statics, whenever $x''_{-a} > x'_{-a}$, we have $BR_a(x''_{-a}) \geq BR_a(x'_{-a})$. Let $\phi_a(x_{-a}) = \max BR_a(x_{-a})$, which is increasing and thus G has NE. Then we can argue $\overline{x} = \sup\{x \in X : x \leq \phi(x)\}$ is the largest NE.

    Application: for $f(p_1; p_2) = (p_1 - c_1)D_1(p_1; p_2)$, if $f(p_1; p_2)$ has single crossing differences, and D is continuous in p, then the firms are playing a game of strategic complements so there is a largest and a smallest NE.

* Equilibrium Comparative Statics
  - Idea: compared with single-agent comparative statics, we are dealing with equilibrium decision outputs now, so we'd like to know what's the influence to NE when some parameter t changes(e.g. marginal cost of one firm).
  - Theorem: suppose that for every player $a \neq \hat{a}$, u_a is continuous in x_a and has single crossing differences in $(x_a; x_{-a})$.; for player $\hat{a}$, $u_{\hat{a}}$ is continuous in $x_{\hat{a}}$ and has single crossing differences in $(x_{\hat{a}}; (x_{-\hat{a}}, t))$. Then the largest NE of the game at t'' is larger than the largest NE of the game at t' whenever $t'' > t'$.

    Proof. It's easy to argue that $\phi_{\hat{a}}(x_{-\hat{a}}, t) = \max BR_{\hat{a}}(x_a, x_{-\hat{a}}, t)$ is increasing in t, thus the largest fixed point $x^{**}(t)$ of $\phi(, t)$ is the largest NE. By the theorem from Increasing Maps, we know the largest fixed point of $\phi(, t)$ is also increasing in t.

Application: for $f(p_1;p_2) = (p_1 - c_1)D_1(p_1;p_2)$, if firm 1 experiences an increase in its marginal cost, since $f(p_1,(p_2,c_1))$ has single crossing differences, we know the largest NE of this game is also increasing. For firm 2, he is better off because

$$f(\overline{p}_2^{**};\overline{p}_1^{**}) \geq f(\underline{p}_2^{**};\overline{p}_1^{**}) \geq f(\underline{p}_2^{**};\underline{p}_1^{**})$$

8 Equilibrium in Finanical Economy

- Financial Assets
  - Portfolio: there are L states of the world tomorrow, and S securities, resulting a payoff matrix D. A portfolio z is a linear combination of assets along with its payoff vector Dz. The Span(D) is called asset span, and if it is a strict subspace of R^L, then the economy has incomplete markets.
  - Endowment: an agent has endowment $\omega \in R_+^{L+1}$, then with a portfolio z, his contingent consumption tomorrow is $\omega_{-0} + Dz$.
- Agent's Problem
  - Utility Maximization: the agent's utility function is the combination of today's utility and discounted tomorrow's utility. The decision x for utility maximization is restricted in a budget set B determined by (q, ω, D). Instead of optimize x directly, we will optimize the portfolio z.
- Financial Economy
  - Asset Equilibrium: consider there is a set of agents A who are in the same economy with their own utility functions and buy or sell the assets with some price q. A price q^* in R^S is an equilibrium price of this financial economy F if
    1. for each agent a, there is z^a such that his utility U^a is maximized
    2. $\sum_{a\in A} z^a = 0$
  - Consumption Equilibrium: if q^* is an equilibrium price of F and let z^a, x^a be agent a's asset and consumption demand respectively, then provided U^a is strongly monotone, we have $\sum_{a\in A} x^a = \sum_{a\in A} \omega^a$.
- Constrained Pareto Optimality
  - Feasibility: the allocation $\{x^a\}_{a\in A}$ is constrained feasible allocation if $\sum_{a\in A} x^a = \sum_{a\in A} \omega^a$ and there is $\{z^a\}_{a\in A}$ such that $\sum_{a\in A} z^a = 0$ with $x_{-0}^a = \omega_{-0}^a + Dz^a$. Basically, it means the allocation cannot increase or decrease the asset and should be characterized by some payoff matrix.
  - Optimality: the allocation $\{x^a\}_{a\in A}$ is constrained Pareto optimal if there does not exist another constrained feasible allocation that is Pareto superior.
  - First Welfare Theorem for Incomplete Markets: if U^a is strongly monotone for all a, then every equilibrium allocation is constrained Pareto optimal. The proof supposes there is a Pareto superior allocation and shows that consumption equilibrium does not hold, which is a contradiction.
- Equilibrium Invariance
  - Idea: there are many possible markets with different payoffs, and if we only care about equilibrium behavior, we can show when the span of payoff matrix keeps unchanged, then the equilibrium does not change in an essential way.

  - Theorem: suppose q^* is an equilibrium price of $F(D)$, z^a is agent a's portfolio, which achieves consumption of x^a, then the economy $F(D)$ with $D' = DK$ for some invertible K, has the equilibrium price q^*K. The equilibrium portfolio of agent a is $K^{-1}z^a$ and his consumption is x^a, which is unchanged.

- Arbitrage

  - Definition: if there is $z \in R^S$ such that $qz \leq 0$ and $Dz \leq 0$ with either inequality strict, we say q admits arbitrage. An equilibrium price cannot admit arbitrage otherwise the utility maximization will be infinite.

  - Theorem: if q admits no arbitrage, then there is $p >> 0$ such that

$$q_s = \sum_{l \in L} p_l d_{ls}$$

    or equivalently $q = pD$.